Multi-sensor Estimation Fusion for Linear Equality Constrained Dynamic Systems

Zhansheng Duan
Center for Information Engineering Science Research
Xi’an Jiaotong University
Xi’an, Shaanxi 710049, China
Email: zduan@uno.edu

X. Rong Li
Department of Electrical Engineering
University of New Orleans
New Orleans, LA 70148, USA
Email: xli@uno.edu

Abstract—The state of many dynamic systems evolve subject to some linear equality constraints. Using different techniques, different state estimators for linear equality constrained dynamic systems have been developed in the literature, including the pseudo measurement method, the projection method, the null space method, and the direct elimination method. However, their extension to the multi-sensor case has not been addressed. In this paper, using the measurement augmentation technique, we first extend all the above four types of estimators to multi-sensor constrained centralized fusion problems. Then the properties of these centralized fusers are analyzed. For multi-sensor constrained distributed fusion, direct application of standard distributed fusion is involved due to the singularity of local estimation mean square error (MSE) matrices. We suggest to use nonstandard distributed fusion by fully considering the difference between constrained and unconstrained state estimation. For multi-sensor constrained distributed fusion, we propose to send in dimension reduced local state estimates which are directly available in both the null space method and the direct elimination method. In this way, the singularity issue of the MSE matrices can be circumvented with saved communication from local sensors to the fusion center. Numerical examples further illustrate the similarities and differences among different estimation fusers.

Keywords: Constrained system, constrained estimation, estimation fusion, centralized fusion, distributed fusion.

I. INTRODUCTION

The state of many dynamic systems evolves subject to some equality constraints. For example, in ground target tracking [1]–[3], if we treat roads as curves without width, the road networks can then be described by equality constraints. In an airport, an aircraft moves on the runways or taxiways. In a quaternion-based attitude estimation problem, the attitude vector must have a unit norm [4]. In a compartmental model with zero net inflow [5], mass is conserved. In undamped mechanical systems, such as one with Hamiltonian dynamics, the energy conservation law holds. Likewise, in electric circuit analysis, Kirchhoff’s current and voltage laws hold.

For linear equality constrained (LEC) state estimation, numerous results are available. For example, the dimensionality reduction method equivalently converts a constrained state estimation problem to a reduced dimension unconstrained one. The equivalence can be achieved by representing part of the state vector as a linear function of the remaining part making use of the deterministic relationships [6], imposed by the linear equality constraint, among components of the state vector. It can also be achieved through null space decomposition as in [7]. With this method, the complexity of the dynamic system to be estimated can be reduced. However, this reduction is not necessarily significant because the computational load of the state estimator is mainly determined by the dimension of the measurement, which is unaltered in the dimensionality reduction method. Another popular method, the projection method (see, e.g., [8]–[10]), projects the unconstrained estimate onto the constraint subspace by applying classical constrained optimization techniques. Specifically in [8], after the unconstrained estimate has been obtained, the problem is formulated as one of weighted least-squares estimation, in which the unconstrained estimate is treated as data and the inverse of its error covariance matrix is used as the weighting matrix. The pseudo measurement method has also been applied to equality constrained state estimation. Its key idea is to treat the equality constraints as pseudo measurements. Thus the LEC state estimation problem is converted into a regular filtering problem with two types of measurements. However, since the pseudo measurements are noise free, the augmented measurement noise will have a singular covariance. Then numerical problems may occur during filtering. Moreover, the increase in the dimension of the augmented measurement will increase the computational complexity of the state estimator. That is probably why the pseudo measurement method is not popular for LEC state estimation. To avoid possible numerical problems caused by the singular covariance of the measurement noise [11] if the matrix inverse is used, the Moore-Penrose (MP) pseudoinverse was used in [12], [13]. Also, to gain insight and analyze this type of estimation problem and to mitigate the MP inverse of a higher dimension with a demanding computational load in a batch form, two equivalent sequential forms were obtained in [12], [13] by following the recursibility of linear minimum MSE (LMMSE) estimation of [14]. It was found that under certain conditions, although equality constraints are indispensable for the evolution of the state, updating by them is redundant for filtering. If there exists model mismatch, however, updating by them is necessary and helpful.

Estimation fusion, or data fusion for estimation, is the problem of how to best utilize useful information contained in multiple sets of data for the purpose of estimating a quantity—
multi-sensor constrained distributed fusion is involved due to local state estimates. nonstandard distributed fusion and send in dimension reduced from local sensors to the fusion center, we propose to use circumvent this singularity issue and save communication constrained state estimators for the single sensor case. However, constrained dynamic systems. In this paper, we consider the multi-
unconstrained dynamic systems should be different for con-
strained systems. In this paper, we consider the multi-
sensor constrained estimation fusion problem. Multi-sensor constrained centralized fusion is easily tackled by applying the measurement augmentation technique to the existing con-
strained state estimators for the single sensor case. However, multi-sensor constrained distributed fusion is involved due to the singularity of the local estimation MSE matrices. To circumvent this singularity issue and save communication from local sensors to the fusion center, we propose to use nonstandard distributed fusion and send in dimension reduced local state estimates.

This paper is organized as follows. Sec. II formulates the problem. Sec. III presents multi-sensor constrained centralized fusion. Sec. IV presents multi-sensor constrained nonstandard distributed fusion. Sec. V provides two numerical examples to illustrate the similarities and differences among the proposed multi-sensor constrained estimation fusers. Sec. VI gives conclusions.

II. PROBLEM FORMULATION

Consider the following typical form of a linear stochastic dynamic system

\[ x_{k+1} = F_k x_k + G_k u_k + w_k \]

where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \), \( E[x_0] = \bar{x}_0 \), \( \text{cov}(x_0) = P_0 \), \( \langle w_k \rangle \) is a zero-mean white noise sequence with \( \text{cov}(w_k) = Q_k \) and independent of \( x_0 \).

It is known in advance that the state of this system satisfies the following linear equality constraint

\[ C_k x_k = d_k \]

where \( C_k \in \mathbb{R}^{m \times n} \) and \( d_k \in \mathbb{R}^m \) are both known, \( C_k \) is of full row rank, and \( m < n \).

Assuming that altogether \( N \) sensors are used to observe the above constrained dynamic system simultaneously as

\[ z_{k}^{(i)} = H_{k}^{(i)} x_k + v_{k}^{(i)}, \ i = 1, 2, \cdots, N \]

where \( z_{k}^{(i)} \in \mathbb{R}^{m_i} \), \( \langle v_{k}^{(i)} \rangle \) is a zero-mean white measurement noise sequence with \( \text{cov}(v_{k}^{(i)}) = R_k^{(i)} \) and independent of \( x_0 \) and \( \langle w_k \rangle \) and independent across sensors.

The problem of interest is to obtain centralized and distributed fusion of the state system \( x_k \) given the measurement \( \{ z_{j}^{(i)} \}^{j=1, \cdots, k}_{i=1, \cdots, N} \) from all sensors.

III. CONstrained Centralized Fusion

As said in [15], multi-sensor centralized fusion is nothing but an estimation problem with distributed data. And the most natural way to handle centralized fusion is to augment the measurements from multiple sensors and then apply existing state estimators. So in the following, we will discuss the applicability of existing single sensor based constrained state estimation to the multi-sensor case through measurement augmentation first.

Let

\[ z_k = \begin{bmatrix} \langle z_k^{(1)} \rangle^\prime \ 
\langle z_k^{(2)} \rangle^\prime \ \cdots \ 
\langle z_k^{(N)} \rangle^\prime \end{bmatrix}^\prime \]

\[ H_k = \begin{bmatrix} \langle H_k^{(1)} \rangle^\prime \ 
\langle H_k^{(2)} \rangle^\prime \ \cdots \ 
\langle H_k^{(N)} \rangle^\prime \end{bmatrix}^\prime \]

\[ v_k = \begin{bmatrix} \langle v_k^{(1)} \rangle^\prime \ 
\langle v_k^{(2)} \rangle^\prime \ \cdots \ 
\langle v_k^{(N)} \rangle^\prime \end{bmatrix}^\prime \]

Then the measurements from multiple sensors can be written in the following compact form

\[ z_k = H_k x_k + v_k \]

where \( \langle v_k \rangle \) is still a zero-mean white measurement noise sequence with \( \text{cov}(v_k) = R_k \) and

\[ R_k = \text{diag}\{ R_k^{(1)}, R_k^{(2)}, \cdots, R_k^{(N)} \} \]

Now the existing single-sensor based constrained state estimators can be easily applied to multi-sensor constrained centralized fusion as follows.

A. Pseudo measurement method

The idea of the pseudo measurement method [8], [12], [13] is to treat the linear equality constraint as an extra noise-free pseudo measurement of the system state as

\[ d_k = C_k x_k \]

where \( d_k \) is the measurement and \( C_k \) is the measurement matrix. Then traditional linear state estimation can be applied.

One cycle of the pseudo measurement method can be summarized as follows.

- Prediction:

\[ \hat{x}_{k|k-1} = F_{k-1} \hat{x}_{k-1|k-1} + G_{k-1} u_{k-1} \]

\[ P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^\prime + Q_{k-1} \]
Update:
\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k(P_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot (z_k - H_k\hat{x}_{k|k-1}) \]
\[ P_{k|k} = P_{k|k-1} - P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot \bar{H}_kP_{k|k-1} \]

where \( A^+ \) stands for the MP inverse of \( A \) and
\[ \bar{z}_k = \begin{bmatrix} z_k \\ d_k \end{bmatrix}, \quad \bar{H}_k = \begin{bmatrix} H_k \\ C_k \end{bmatrix}, \quad \bar{R}_k = \begin{bmatrix} R_k & 0 \\ 0 & 0 \end{bmatrix} \]

To further reduce the computational complexity of the above constrained state estimator, two equivalent sequential forms for the update step were proposed in [12], [13] as follows.

**Form 1:**
- Update by the physical measurement:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot (z_k - H_k\hat{x}_{k|k-1}) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot H_kP_{k|k-1} \]
- Update by the pseudo measurement:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + (d_k - C_k\hat{x}_{k|k-1})) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + C_kP_{k|k-1}) \]

**Form 2:**
- Update by the pseudo measurement:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + d_k - C_k\hat{x}_{k|k-1}) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + C_kP_{k|k-1}) \]
- Update by the physical measurement:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot (z_k - H_k\hat{x}_{k|k-1}) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot H_kP_{k|k-1} \]

**B. Projection method**

The idea of the projection method [8] is to project the unconstrained state estimate \( \hat{x}_{n|k} \) onto the constraint subspace through the following constrained quadratic programming
\[ \hat{x}_{k|k} = \arg \min_{x_k} (x_k - \hat{x}_{k|k})^T(P_{n|k}^{-1})^{-1}(x_k - \hat{x}_{k|k}) \]
s.t. \( C_kx_k = d_k \)

where \( P_{n|k} \) is the MSE matrix of \( \hat{x}_{n|k} \).

One cycle of the projection method can be summarized as follows.
- Prediction: same as in the pseudo measurement method.
- Unconstrained update:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot (z_k - H_k\hat{x}_{k|k-1}) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}H_k^T(\bar{H}_kP_{k|k-1}H_k^T + \bar{R}_k)^{-1} \cdot H_kP_{k|k-1} \]
- Project the unconstrained estimate onto the constraint subspace:
  \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + (d_k - C_k\hat{x}_{k|k-1})) \]
  \[ P_{k|k} = P_{k|k-1} - P_{k|k-1}C_k^T(\bar{C}_kP_{k|k-1}C_k^T + C_kP_{k|k-1}) \]

**C. Null space method**

The idea of the null space method [7] is to represent the general solution of the constraint equation as a summation of a deterministic special solution and a stochastic zero solution through null space decomposition. In this way, the original constrained state estimation can be reduced to unconstrained state estimation about the zero solution part.

One specific way to fulfill the above goal is to denote the QR factorization of \( C_k^T \) as
\[ C_k^T = \begin{bmatrix} Q_1^T & Q_2^T \end{bmatrix} \begin{bmatrix} R_{11}^T \\ 0 \end{bmatrix} \]
where \( Q_1 \in \mathbb{R}^{n \times m} \) and \( Q_2^T \in \mathbb{R}^{m \times (n-m)} \) are orthogonal matrices, and \( R_{11} \in \mathbb{R}^{m \times m} \) is nonsingular and upper triangular.

Then we have the following dimension reduced unconstrained dynamic system
\[ r_{k+1} = (Q_2^T)^T(F_kQ_2^T + (Q_2^T)^TK_kv_k) \]
\[ z_k = H_kQ_2^Tr_k + H_kx_k + v_k \]
and
\[ x_k = x_k^d + Q_1^TR_{11}^{-1}d_k \]

is a deterministic special solution of the constraint equation.

Given that
\[ \hat{x}_{0|0} = \bar{x}_0, \quad P_{0|0} = P_0 \]
we have
\[ \hat{r}_{0|0} = (Q_2^T)^T(\hat{x}_{0|0} - x_0^d) \]
\[ P_{0|0} = (Q_2^T)^TP_0Q_2 \]

One cycle of the null space method can be summarized as follows.
- Prediction:
  \[ \hat{x}_{k|k-1} = (Q_1^T)^TF_{k-1}Q_2^T - (Q_1^T)^TF_{k-1}Q_2^T \]
  \[ + (Q_2^T)^T(F_kx_k^d + G_ku_k) \]
  \[ + (Q_2^T)^TQ_2^Tr_k \]
  \[ + (Q_2^T)^TQ_2^T \]
  \[ \hat{r}_{k|k-1} = (Q_1^T)^TF_{k-1}Q_2^T - (Q_1^T)^TF_{k-1}Q_2^T \]
  \[ + (Q_2^T)^TQ_2^T \]
  \[ P_{k|k-1} = (Q_1^T)^TP_{k-1}Q_2 \]
  \[ \hat{x}_{k|k} = x_k^d + Q_2^Tr_{k|k-1} \]
  \[ P_{k|k} = Q_2^TP_{k|k-1}(Q_2^T)^T \]
which leads to the following unconstrained dynamic subsystem

\[ \dot{r}_{k|k} = \dot{r}_{k|k-1} + P_{r|k-1}^k (H_k Q_k^2)^{1/2} \]

\[ \cdot (H_k Q_k^2 P_{r|k-1}^k (H_k Q_k^2)^{1/2} + R_k)^{-1} \]

\[ (z_k - H_k \hat{x}_{k|k-1} - H_k x_k^d) \]

\[ P_{r|k}^k = P_{r|k-1}^k - P_{r|k-1}^k (H_k Q_k^2)^{1/2} \]

\[ \cdot (H_k Q_k^2 P_{r|k-1}^k (H_k Q_k^2)^{1/2} + R_k)^{-1} H_k Q_k^2 P_{r|k-1}^k \]

\[ \hat{x}_{k|k} = x_d^k + Q_k^2 \hat{r}_{k|k} \]

\[ P_{k|k} = Q_k^2 P_{r|k}^k (Q_k^2)^{1/2} \]

(7) (8)

\[ D. \ Direct \ elimination \ method \]

The idea of the direct elimination method [6] is to represent part (dependent part) of the state vector as a linear function of the remaining part (independent part) of the state vector. In this way, the dependent part can be eliminated from the original state transition equation and we only need to estimate the unconstrained independent part.

In the direct elimination method, without loss of generality, assume that the components of \( x_k \) have been reshuffled such that \( C_k^1 \) is nonsingular for

\[ C_k = \begin{bmatrix} C_k^1 & C_k^2 \end{bmatrix} \]

Then the state transition equation, the constraint equation and the measurement matrix can be rewritten accordingly as

\[ \begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} F_{k}^{11} & F_{k}^{12} \\ F_{k}^{21} & F_{k}^{22} \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} G_k^1 \\ G_k^2 \end{bmatrix} \hat{u}_k + \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix} \]

\[ C_k \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = d_k \]

\[ H_k = \begin{bmatrix} H_k^1 \\ H_k^2 \end{bmatrix} \]

which leads to the following unconstrained dynamic subsystem

\[ x_{k+1}^2 = (F_{k}^{22} - F_{k}^{21} A_k) x_{k}^2 + F_{k}^{21} (C_k^1)^{-1} d_k + G_k^2 u_k + w_{k}^2 \]

\[ z_k = (H_k^2 - H_k^1 A_k) x_{k}^1 + H_k^1 (C_k^1)^{-1} d_k + \hat{u}_k \]

and

\[ x_k^1 = (C_k^1)^{-1} d_k - A_k x_{k}^2 \]

where

\[ A_k = (C_k^1)^{-1} C_k^2 \]

One cycle of the direct elimination method can be summarized as follows.

- **Prediction:**

\[ \begin{array}{l}
\hat{x}_{k|k-1}^2 = (F_{k}^{22} - F_{k}^{21} A_k) \hat{x}_{k-1|k-1}^2 + F_{k}^{21} (C_k^1)^{-1} d_{k-1} + G_k^2 u_{k-1} \\
\hat{P}_{k|k-1}^2 = (F_{k}^{22} - F_{k}^{21} A_k) \hat{P}_{k-1|k-1}^2 (F_{k}^{22} - F_{k}^{21} A_k)^{1/2} + Q_{k-1}^2 \\
\end{array} \]

(9)

\[ \begin{array}{l}
\hat{P}_{k|k-1}^1 = A_k \hat{P}_{k|k-1}^2 A_k^T - A_k \hat{P}_{k|k-1}^2 A_k^T + A_k \hat{P}_{k|k-1}^2 A_k^T \end{array} \]

(10)

\[ \begin{array}{l}
\hat{x}_{k|k-1}^1 = (C_k^1)^{-1} d_k - A_k \hat{x}_{k|k-1}^2 \\
\hat{P}_{k|k-1}^1 = A_k \hat{P}_{k|k-1}^2 A_k^T - A_k \hat{P}_{k|k-1}^2 A_k^T + A_k \hat{P}_{k|k-1}^2 A_k^T \\
\end{array} \]

(11) (12)

\[ \begin{array}{l}
P_{k|k-1}^1 = \begin{bmatrix} A_k \hat{P}_{k|k-1}^2 A_k^T & -A_k \hat{P}_{k|k-1}^2 A_k^T \\
-P_{k|k-1}^2 A_k^T & P_{k|k-1}^2 \end{bmatrix} \end{array} \]

(13) (14)

where

\[ Q_k^2 = \text{cov}(\hat{w}_{k}^2) \]

- **Update:**

\[ \begin{array}{l}
\hat{r}_{k|k} = \hat{r}_{k|k-1} + P_{r|k-1}^k (H_k^2 - H_k^1 A_k)^{1/2} \\
\cdot (H_k^2 - H_k^1 A_k)^{-1} (H_k^2 - H_k^1 A_k)^{1/2} \cdot (z_k - (H_k^2 - H_k^1 A_k) \hat{x}_{k|k-1}^2 - H_k^1 (C_k^1)^{-1} d_k) \\
P_{r|k}^k = P_{r|k-1}^k - P_{r|k-1}^k (H_k^2 - H_k^1 A_k)^{1/2} \\
\cdot (H_k^2 - H_k^1 A_k)^{-1} (H_k^2 - H_k^1 A_k)^{1/2} \\
\cdot (H_k^2 - H_k^1 A_k)^{-1} P_{r|k-1}^k \end{array} \]

(15)

\[ \begin{array}{l}
P_{k|k}^1 = A_k \hat{P}_{k|k-1}^2 A_k^T - A_k \hat{P}_{k|k-1}^2 A_k^T + A_k \hat{P}_{k|k-1}^2 A_k^T \\
P_{k|k}^2 = P_{k|k-1}^2 - P_{k|k-1}^2 A_k^T A_k \hat{P}_{k|k-1}^2 A_k^T + A_k \hat{P}_{k|k-1}^2 A_k^T + A_k \hat{P}_{k|k-1}^2 A_k^T \\
\end{array} \]

(16) (17) (18)

Now let us summarize the properties of the above four constrained centralized fusion methods.

- The four methods have the same fusion performance although the underlying fusion rules to get the fused estimate are different, which lead to slight differences in computational complexity.

- Both the predicted and updated fused estimates satisfy the given constraint. That is,

\[ C_k \hat{x}_{k|k-1} = d_k, \ C_k \hat{x}_{k|k} = d_k \]

For the pseudo measurement method, the update by the pseudo measurement is redundant. For the projection method, the projection of the unconstrained estimate onto the constraint subspace is also redundant. That is,

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1}, \ \hat{x}_{k|k} = \hat{x}_{k|k-1}, \ \hat{x}_{k|k} = \hat{x}_{k|k-1}, \ P_{k|k} = P_{k|k-1} \]

- It follows that

\[ C_k P_{k|k-1} C_k^T = 0, \ C_k P_{k|k-1} C_k^T = 0 \]

This is because from the direct elimination method, we have

\[ C_k P_{k|k-1} C_k^T = \begin{bmatrix} C_k^1 & C_k^2 \end{bmatrix} \begin{bmatrix} A_k P_{k|k-1}^2 A_k^T - A_k P_{k|k-1}^2 A_k^T \\
-P_{k|k-1}^2 A_k^T & P_{k|k-1}^2 \end{bmatrix} \]

\[ = C_k^1 A_k P_{k|k-1}^2 A_k^T (C_k^1)^{-1} - C_k^1 A_k P_{k|k-1}^2 (C_k^1)^{-1} \\
- C_k^1 P_{k|k-1}^2 A_k^T (C_k^1)^{-1} + C_k^2 P_{k|k-1}^2 (C_k^1)^{-1} \\
= C_k^1 (C_k^1)^{-1} C_k^2 P_{k|k-1}^2 (C_k^1)^{-1} - C_k^1 (C_k^1)^{-1} \\
- C_k^2 P_{k|k-1}^2 (C_k^1)^{-1} + C_k^2 P_{k|k-1}^2 \]

\[ = 0 \]
which simply means that the innovation associated with the pseudo measurement does not bring any new information to the constrained state estimation. Similarly we can also prove $C_k P_{k|k} C_k^T = 0$.

- It also follows that both $P_{k|k-1}$ and $P_k$ are singular. This is because from the direct elimination method, we have

$$x_{k|k-1}^2 = -A_k x_{k|k-1}^2, \quad x_k^2 = -A_k x_k^2$$

IV. CONSTRAINED DISTRIBUTED FUSION

For multi-sensor constrained distributed fusion, we need to figure out what kind of local constrained estimation to use first. From the discussion above, we have at least four choices for local estimation—the pseudo measurement method, the projection method, the null space method, and the direct elimination method. For constrained centralized fusion, all four of them can be applied. They have the same fusion performance except some subtle difference in computational complexity. For constrained distributed fusion, although we can still use all four of them as local estimators, they will lead to different difficulties in the design of the fusion rule at the fusion center as explained below. For constrained distributed fusion, the design of the fusion rule and the selection of local estimators should be considered jointly.

From the above, we know that it does not matter which one of the four methods is used as local estimators, the predicted MSE matrix $P_{k|k-1}$ and the updated MSE matrix $P_k$ are always singular at any local sensor $i$. The singularity prevents the use of most existing distributed fusion rules at the fusion center, e.g., the simple convex combination method [16], [17] and the information matrix fusion rule [18]. Does this mean that for constrained distributed fusion, we have to resort to some computationally intensive fusion rules, e.g., using MP inverse to handle singularity? The answer is no because the singularity in constrained state estimation is well structured. By appropriately using the structure governing the singularity of the predicted and updated MSE matrices, we can still use the existing distributed fusion rules, e.g., the simple convex combination method and the information matrix fusion rule, with some minor changes as shown below.

The above discussion still follows the standard distributed fusion in which the local estimates are sent to the fusion center. Due to the singularity of the predicted and updated MSE matrices, we encounter major difficulty in the design of the fusion rule. Then how about nonstandard distributed fusion in which not the local estimates are sent to the fusion center from local sensors (e.g., a linear transformation of the local measurement data)? From [11], we know that some nonstandard distributed fusion rules can not only have the optimal performance but also save communication from local sensors to the fusion center and sometimes even increase numerical robustness. In the following, we will mainly consider nonstandard distributed fusion rules for the multi-sensor constrained distributed fusion problem.

If we use the null space method at local sensors, other than the estimates $\hat{x}_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}, \hat{z}_{k|k}^{(i)}, P_{k|k}^{(i)}$ of the full state $x_k$, we also have the estimates $\hat{x}_{k|k-1}^{(i)} P_{k|k-1}^{r,(i)}, \hat{x}_{k|k}^{(i)} P_{k|k}^{r,(i)}$ of the dimension reduced state $r_k$. Then a natural nonstandard distributed fusion is to send in $\hat{r}_{k|k-1}^{(i)}, P_{k|k-1}^{r,(i)}$ and $\hat{z}_{k|k}^{(i)} P_{k|k}^{r,(i)}$. This has two advantages over the standard distributed fusion. First, $P_{k|k-1}^{r,(i)}$ and $P_{k|k}^{r,(i)}$ are usually nonsingular. This can help get rid of the singularity issue discussed above for standard distributed fusion. Second, sending in $\hat{x}_{k|k-1}^{(i)}, P_{k|k-1}^{r,(i)}$ and $\hat{z}_{k|k}^{(i)} P_{k|k}^{r,(i)}$ can save communication since the dimension of $r_k$ is also $n-m$, which is smaller than the dimension $n$ of the system state $x_k$.

For illustration, here we only introduce how to apply the information matrix fusion rule for the distributed fusion of the dimension reduced state $r_k$. The other existing distributed fusion rules, e.g., the simple convex combination rules, can be applied similarly. One cycle of the information matrix fusion rule can be summarized as follows.

- Fused prediction of $r_k$: use (3) and (4)
- Fused prediction of $x_k$: use (5) and (6)
- Fused update of $r_k$

$$
(P_{k|k}^{r})^{-1} = \left( P_{k|k-1}^{r} \right)^{-1} + \sum_{i=1}^{N} \left( (P_{k|k}^{r,(i)})^{-1} - (P_{k|k-1}^{r,(i)})^{-1} \right)
$$

$$
\left( P_{k|k}^{r} \right)^{-1} \hat{r}_{k|k-1} = \left( P_{k|k-1}^{r} \right)^{-1} \hat{r}_{k|k-1} + \sum_{i=1}^{N} \left( (P_{k|k}^{r,(i)})^{-1} - (P_{k|k-1}^{r,(i)})^{-1} \right) \hat{r}_{k|k-1}^{(i)}
$$

- Fused update of $x_k$: use (7) and (8)

Similarly we can also use the direct elimination method at local sensors. Other than sending in the estimates $\hat{x}_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}, \hat{z}_{k|k}^{(i)}, P_{k|k}^{(i)}$ of the full state $x_k$, we can just send in the estimates $\hat{x}_{k|k-1}^{2,(i)}, P_{k|k-1}^{2,(i)}, \hat{z}_{k|k}^{2,(i)}, P_{k|k}^{2,(i)}$ of the sub-state vector $x_k^2$. The two advantages of doing so over the standard distributed fusion are the same as the use of the null space method at local sensors. First, $P_{k|k-1}^{2,(i)}$ and $P_{k|k}^{2,(i)}$ are usually nonsingular. Second, sending in $\hat{x}_{k|k-1}^{2,(i)}, P_{k|k-1}^{2,(i)}, \hat{z}_{k|k}^{2,(i)}, P_{k|k}^{2,(i)}$ instead of $\hat{x}_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}, \hat{z}_{k|k}^{(i)}, P_{k|k}^{(i)}$ can save communication from local sensors to the fusion center since the dimension $n-m$ of $x_k^2$ is smaller than the dimension $n$ of the system state $x_k$. One cycle of the information matrix fusion rule can be summarized as follows.

- Fused prediction of $x_k^2$: use (9) and (10)
- Fused prediction of $x_k$: use (11)–(14)
the desired linear equality constraint can be described as

\[ \text{line road is dimensional space.} \]

\[ \text{It is known that the slope of the straight velocity motion of a vehicle along a straight line road in a two dimensional space.} \]

\[ \text{As discussed in [20], if we system and they should take some structure governed by the above linear equality constraint.} \]

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\[ \text{Consider the following dynamic system, which describes the nearly constant velocity motion of a vehicle in a two dimensional space} \]

\[ x_{k+1} = F_k x_k + G_k w_k \]

where

\[ x_k = [x_k \ x_k \ y_k \ y_k] \]

\[ x_0 \sim \mathcal{N} (\bar{x}_0, P_0), \ w_k \sim \mathcal{N} (0, Q_k) \]

\[ F_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ G_k = \begin{bmatrix} T^2/2 & 0 \\ 0 & T \\ 0 & T^2/2 \\ 0 & T \end{bmatrix}, \ T = 2 \]

It is expected to use the above discrete white noise acceleration (DWNA) model [19] to describe the nearly constant velocity motion of a vehicle along a straight line road in a two dimensional space. It is known that the slope of the straight line road is \( a = \tan (\pi/3) \) and its \( y \)-intercept is \( b = 10 \). Then the desired linear equality constraint can be described as

\[ C_k x_k = d_k \]

where

\[ C_k = \begin{bmatrix} a & 0 & -1 & 0 \\ 0 & a & 0 & -1 \end{bmatrix}, \ d_k = \begin{bmatrix} -b \\ 0 \end{bmatrix} \]

To guarantee that the system state of the DWNA model satisfies this desired linear equality constraint, as discussed in [6, 20], \( x_0, w_k \) and the system parameters \( \bar{x}_0, P_0, Q_k \) can not be arbitrarily designed as for an unconstrained dynamic system and they should take some structure governed by the above linear equality constraint. As discussed in [20], if we specify them as

\[ x_0 = \begin{bmatrix} \frac{1}{a} (y_0 \ y_0) & + d_k^T \ y_0 \ y_0 \ y_0 \ y_0 \end{bmatrix}^T \]

\[ \bar{x}_0 = \begin{bmatrix} \frac{1}{a} (260 \ 60) & + d_k^T \ 260 \ 60 \end{bmatrix}^T \]

\[ P_0 = \frac{1}{a} \text{diag}(625, 100) \]

\[ w_k = \frac{1}{a} \text{diag}(0.2^2, 0.2^2), \quad w_k \sim \mathcal{N}(0, 0.2^2) \]

\[ Q_k = \frac{1}{a} \text{diag}(0.2^2, 0.2^2) \]

\[ \hat{x}_0 = \begin{bmatrix} 260 & 60 & 260 & 60 \end{bmatrix}^T \]

\[ P_0 = \text{diag}(625, 100, 625, 100) \]

\[ w_k = \frac{1}{a} \text{diag}(0.2^2, 0.2^2), \quad w_k \sim \mathcal{N}(0, 0.2^2) \]

\[ Q_k = \frac{1}{a} \text{diag}(0.2^2, 0.2^2) \]

\[ \text{the desired linear equality constraint can be guaranteed to be satisfied.} \]

\[ \text{Assuming that two sensors are used to observe the constrained motion of the vehicle on the road as} \]

\[ x_k^{(i)} = H_k^{(i)} x_k + v_k^{(i)}, \ i = 1, 2 \]

where

\[ H_k^{(1)} = H_k^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]

\[ R_k^{(1)} = \text{diag}(20^2, 20^2), \ R_k^{(2)} = \text{diag}(30^2, 30^2) \]

In the following, we compare the estimation performance of the constrained centralized fuser using the pseudo measurement method (pseudo), the projection method (projection), the null space method (null) and the direct elimination method (direct) in terms of RMS position and velocity errors over 200 Monte Carlo runs.

\[ \text{A. Case 1} \]

In this case, all four centralized fusers are optimally initialized by

\[ \hat{x}_{0|0} = \bar{x}_0, \ P_{0|0} = P_0 \]

and the process noise covariance matrix used by them is \( Q_k \). That is, the system parameters used by all four centralized fusers match the ones used to generate the ground truth.

Figs. 1 and 2 show the estimation performance of all four centralized fusers.

It can be easily seen that the error curves for all four centralized fusers overlap with each other. This means that they have exactly the same estimation performance, as discussed above.

\[ \text{B. Case 2} \]

In this case, all four centralized fusers are initialized by

\[ \hat{x}_{0|0} = \begin{bmatrix} 260 & 60 & 260 & 60 \end{bmatrix}^T \]

\[ P_{0|0} = \text{diag}(625, 100, 625, 100) \]

and the process noise covariance matrix used by them is

\[ Q_k^{(i)} = \text{diag}(0.2^2, 0.2^2) \]
That is, the system parameters used by all four centralized fusers do not match the ones used to generate the ground truth.

Figs. 3 and 4 show the estimation performance of all four centralized fusers.

It can be easily seen that the estimation performance differs among the four centralized fusers. During the transient, the error curves of the centralized fusers using the pseudo measurement method, the projection method, and the null space method overlap with each other. This means that these three centralized fusers still have the same performance when the system parameters used by the fusers are mismatched with those used for the ground truth generation in this case. It can also be seen that the centralized fuser using the direct elimination method beats the other three slightly during the transient. At the steady state, they have almost no difference. So in view of this and considering that the direct elimination method has reduced communication and can help circumvent the singularity issue in distributed fusion, it seems that we should promote the use of the direct elimination method for multi-sensor constrained estimation fusion.

VI. Conclusions

Using different techniques, state estimation for linear equality constrained dynamic systems is well studied, including the pseudo measurement method, the projection method, the null space method, and the direct elimination method. In this paper, we consider extension of all these constrained state estimators to multi-sensor constrained estimation fusion. Using measurement augmentation, we extend them to multi-sensor constrained centralized fusion and analyze their similarities and differences. Their direct extension to multi-sensor distributed fusion is involved due to the singularity of the local estimation MSE matrices. To circumvent this singularity issue and save communication from local sensors to the fusion center, we propose to send in dimension reduced local estimates to the fusion center, which are directly available in the null space method and the direct elimination method.

REFERENCES


