A Robust and Efficient Particle Filter for Target Tracking with Spatial Constraints

Viktor Pirard  
Saab AB  
Järfälla, Sweden  
viktor.pirard@saabgroup.com

Egils Sviestins  
Saab AB  
Järfälla, Sweden  
egils.sviestins@saabgroup.com

Abstract—This paper addresses the problem of including the non-standard information given by hard constraints in a particle filter for target tracking. The methods previously available for including this information work well in most cases. However, there are situations when the performance of these methods can deteriorate. To this end, a new method, built on proposal distributions adapted to the constraints, is developed. Moreover, the derivation of the Rao-Blackwellized Particle Filter is extended to the case of hard constraints. Both these techniques are combined and demonstrated by two illustrative simulations, showing the potential of the developed methods to handle spatial constraints given by road and coastline information.

Keywords—Particle filter, constrained state estimation, context data, Rao-Blackwellized Particle Filter, proposal distribution

I. INTRODUCTION

The focus of this article is the problem of target tracking in the presence of constraints. Tracking has been extensively studied in the literature, see e.g. [1] and [2] for some classical references and an in-depth treatment of the subject. The core of the problem is to do statistical inference about the target state in an on-line recursive manner. To this end, Bayesian recursions provide optimal filtering solutions, see e.g. [3]. Classically, the Kalman filter and its derivatives have been the method of choice to solve the target tracking problem. The Kalman filter solves the Bayesian recursions analytically for linear problems with Gaussian noise. However, many real-life problems do not meet these criteria, and there is then no general closed form solution to the Bayesian recursion. In such cases different approximations can be considered, e.g. the particle filter (PF) [4], [5] which approximates the solutions to the Bayesian recursions.

When tracking real-world targets there are often sources other than sensor reports available, namely ‘knowledge’ in various shapes. For instance it could be road maps, coastline information, terrain information, or speed limitations. This paper looks into the inclusion of knowledge that can be modeled as (hard) constraints on target states $x_k$. Examples are road networks (if the target is restricted to the road), geographic boundaries and strict speed limitations. This paper is limited to the case of a single target and sensors giving no false or missed detections; hence the otherwise very important data association problem (e.g. [2]) can here be ignored.

The problem of tracking in the presence of constraints has been studied by several authors, giving rise to different approaches as will be shown in section III. The contribution of this paper is to suggest two, as far as we know new, techniques that can be used separately or jointly to make the tracking efficient and robust:

1) Rao-Blackwellized Particle Filter (RBPF) in conjunction with constraints.
2) PF with proposal functions robustly adapted to the constraints.

These techniques will be developed in the following sections, and their properties will be shown by some simulations. But first we will give a somewhat stricter problem definition.

II. PROBLEM DEFINITION

Consider a target with state $x_k \in \mathbb{R}^{n_x}$ at time $k$, obeying the dynamics

$$x_k = f(x_{k-1}) + v_k.$$  \hfill (1)

Here $f(\cdot)$ is a possibly non-linear function and $v_k \sim \mathcal{N}(0, Q_k)$ is white Gaussian process noise. The target also obeys the constraints $C_k$ through

$$x_k \in C_k.$$  \hfill (2)

Here the constrained set is defined as

$$C_k \subset \mathbb{R}^{n_x}.$$  \hfill (3)

Assume further that the target is observed through the measurement model

$$y_k = h(x_k) + e_k,$$  \hfill (4)

where the measurement noise $e_k \sim \mathcal{N}(0, R_k)$ is white and Gaussian. The overarching problem is to find the posterior density $p(x_k | Y^k, C^k)$ at time $k$ given the posterior density at time $k-1$, $p(x_{k-1} | Y^{k-1}, C^{k-1})$, with $Y^k = \{y_1, y_2, \ldots, y_k\}$ and $C^k = \{C_1, C_2, \ldots, C_k\}$ in a robust and efficient manner. The techniques to this end that will be developed here are

1) The RBPF, or marginalized PF, for cases where the state can be split into a conditionally linear and a non-linear part, and where the constraints only apply to the non-linear part.
2) Designing a proposal distribution that ensures that particles are drawn from allowed regions.

The motivation for (1) is that, when applicable, it can reduce the required number of particles. The motivation for (2) is that the common choice of proposal distribution, that is $q(x_k | x_{k-1}, y_k) = p(x_k | x_{k-1})$, can be quite inefficient in cases where a large portion of the proposed particles falls in forbidden regions.
III. RELATED WORK

There are several different approaches taken in the literature to the inclusion of $C_k$ in the estimation of the target state. In this section, we present some common approaches and give related references.

A. Dynamical modeling

When modeling the knowledge of a hard constraint, one may consider the possibility that the target behaves differently depending on the relation to the constraint, e.g. the distance to it or the angle with which is approaching it. Do the same equations of motion describe the target motion with and without the knowledge of the constraint? One approach is to include the knowledge of hard constraints in the dynamical model describing the motion of the target. Much work has been done along these lines for road-bound targets [6], [7], [8], [9]. These papers have in common that they describe the motion in a road-coordinate system that naturally models the constraints, thereby including the constraint in the equations of motion. Roads are considered one-dimensional (no width), and the dynamics describe motion along the road.

For more general constraints, given by coastlines for instance, it is in most cases not possible to include the constraints in the dynamics by a change of coordinates. Here, the dynamics should instead model the behavior of the target in relation to the constraints. As far as we are aware, there has not been much work following this idea. One paper that has taken this approach is [10], where a dynamical model that models the target motion having constant speed (not velocity) and steering much work following this idea. A Related reference.

B. Field modeling

Another approach, similar to the dynamical modeling approach, is to model the impact of the constraint as a field, e.g. distance field or potential field. The dynamics of the target is then adapted to take into account the influence of the field. In [12] inaccessible areas are considered to have a repulsive potential, sending the target away from them. In [13], the target motion is instead influenced by a distance field, with different repulsive forces for different parts of the terrain, another related reference is [14].

C. Optimization formulation

In [15], it is argued that including the constraints on the state via a generalized likelihood in a particle filter (as will be discussed later), does not provide a robust method for including constraints as there is no guarantee that any particles will contribute to the estimation of the posterior distribution. Instead, the Bayesian estimation problem is written as a constrained optimization problem. A couple of different formulations are presented, with the focus of creating a more robust and efficient way of including state constraints. The approach requires a constrained optimization problem to be solved for each particle, giving rise to a high load per particle.

D. Modeling in the prediction step

One of the first articles that suggested the inclusion of a hard constraint in a particle filter was [16], where the constraints were given by a flight envelope (accelerations and speed limitations). The Bayesian filter recursions for including the information of the constraints were derived as

$$p(x_k|Y^k, C_k) \propto \frac{p(y_k|x_k)p(x_k|Y^{k-1}, C_k)}{p(y_k|Y^{k-1}, C_k)}.$$ (5)

The predictive density $p(x_k|C_k, Y^{k-1})$ is given by

$$p(x_k|C_k, Y^{k-1}) = \int p(x_k|x_{k-1}, C_k)p(x_{k-1}|Y^{k-1}, C_k) dx_{k-1}.$$ (6)

The density $p(x_k|x_{k-1}, C_k)$ is given as the truncated dynamics

$$p(x_k|x_{k-1}, C_k) = \left\{ \begin{array}{ll} \int_{C_k} p(x_k|x_{k-1}) dx_k, & x_k \in C_k \\ 0, & x_k \notin C_k \end{array} \right.$$ (7)

Particle filter approximation: There does not exist a general analytical solution to (5), a particle filter can be used to approximate the solution instead. A proposal distribution $q(x_k|x_{k-1}, y_k, C_k)$, that is easy to sample from is introduced. The particle filter approximation of (5) really estimates the joint posterior $p(X^k|Y^k, C_k)$ of the whole trajectory $X^k = \{x_1, x_2, ..., x_k \}$. Since we are interested in estimates of $p(x_k|Y^k, C_k)$, an approximation of this density is

$$p(x_k|Y^k, C_k) \approx \sum_{i=1}^N w_k^i \delta(x_k - x_k^i)$$ (8)

$$w_k^i \propto \frac{p(y_k|x_k^i)p(x_k^i|x_{k-1}^i, C_k)}{q(x_k^i|x_{k-1}^i, y_k, C_k)} w_{k-1}^i$$ (9)

When a hard constraint is modeled in the dynamics as above, the approach often seen taken in literature is to take the proposal distribution as the constrained prior, i.e. $q(x_k|x_{k-1}, y_k, C_k) = p(x_k|x_{k-1}, C_k)$, simplifying the weight calculation in (9). To sample from the constrained dynamics $p(x_k|x_{k-1}, C_k)$, one general approach suggested in [16], is called rejection sampling. The technique is simple and works by drawing samples from the unconstrained distribution $p(x_k|x_{k-1}^i)$, accepting the sample if it belongs to the constrained set, redrawing if not. Since the difference between $p(x_k|x_{k-1})$ and $p(x_k|x_{k-1}, C_k)$ is only a scale factor if $x_k \in C_k$, samples drawn from $p(x_k|x_{k-1})$ will be distributed as $p(x_k|x_{k-1}, C_k)$ if $x_k \in C_k$. The downside of the method is that the time complexity of the sampling algorithm can be very high, limiting its use in an online application.1

1Rejection sampling is not the only way of sampling a truncated distribution. In some cases, it is possible to draw samples more efficiently from a truncated distribution. For cases when it is possible to calculate the cumulative distribution function (CDF), one can use the inverse CDF method to draw samples directly from the truncated transition dynamics as was noted in [17]. For the case of a multivariate normal distribution with linear inequality constraints, it is possible to use the Gibbs sampler from [18] to draw values directly from the truncated distribution.
E. Modeling in the update step

Another way of including the information in a constraint is to define a generalized likelihood
\[ p(C_k|x_k) = \begin{cases} 1, & x_k \in C_k \\ 0, & \text{otherwise} \end{cases}. \] (10)

The Bayesian recursion for including the generalized likelihood is given by
\[ p(x_k|Y^k, C_k) = \frac{p(y_k|x_k)p(C_k|x_k)p(x_k|Y^{k-1}, C^{k-1})}{p(y_k|Y^{k-1}, C_k)p(C_k|C^{k-1})}, \] (11)
where
\[ p(x_k|Y_{k-1}, C_{k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|Y_{k-1}, C_{k-1}) \, dx_{k-1}. \] (12)

Particle filter approximation: The particle filter approximation of the solution to (11) is given by (8) where the weights are now given by
\[ w^i_k \propto p(y_k|x_k)p(C_k|x_k)p(x_k|x_{k-1}) q(x_k|x_{k-1}, Y_k, C_k) w^i_{k-1}, \] (13)
and the particle \( x^i_k, \) are drawn from the proposal distribution.

As far as the authors know, the modeling of a hard constraint in the update step has only been used with the unconstrained prior as proposal distribution, that is \( q(x_k|x_{k-1}, Y_k, C_k) = p(x_k|x_{k-1}). \) This simplifies the weight calculation in (13) to
\[ w_k \propto p(y_k|x_k)p(C_k|x_k)w_{k-1}. \] (14)

It was proven in [19] that the posterior distribution given by the modeling in (5) and (11) are equal, an important result for an on-line application. The upside of the modeling in section III-D and using the constrained prior as the proposal distribution is that it guarantees that all the drawn particles represent admissible states. The downside of the approach is that it can be computationally demanding to sample from the constrained prior. The upside of doing the modeling according to (11) and using the unconstrained prior as proposal distribution, is that the distribution is easy to sample. The downside of using the unconstrained prior as proposal distribution is that there is no guarantee that any of the proposed particles will contribute to the estimation of posterior distribution, i.e. have a weight greater than zero.

IV. CONSTRAINED RAO-BLACKWELLIZED PARTICLE FILTER

Assume that the state vector of the system given in section II can be split into \( x_k = (x_n^k, x_l^k)^T, \) where \( x_n^k \) denotes the non-linear states and \( x_l^k \) the conditionally linear Gaussian states (conditioned on the nonlinear trajectory \( X^n_k = \{ x_n^k, x_n^{k+1}, \ldots, x_n^N \} \) and the measurements \( Y^k \)). The conditionally linear Gaussian states are assumed independent of the constraints (for example, this may be the case when the linear part are velocity components and the constraints are purely spatial). This system can be written as
\[ x_n^k = f^n(x_n^{k-1}) + A^n(x_n^{k-1})x_l^k + v^n_k \] (15)
\[ x_l^k = f^l(x_l^{k-1}) + A^l(x_l^{k-1})x_l^k + v^l_k \] (16)
\[ y_k = h(x_n^k) + (x_l^k) + e_k \] (17)
\[ v_k = (v^n_k, v^l_k)^T \sim \mathcal{N}(0, Q_k), \] (18)
where
\[ Q_k = \begin{pmatrix} Q^n_k & Q^{nl}_k \\ Q^{nl}_k & Q^l_k \end{pmatrix}. \]

The system (15) - (18), contains a conditionally linear and Gaussian substructure. Such a substructure can be used to create a filter with a lower or equal variance and higher efficiency [20], leading to the Rao-Blackwellized Particle Filter (RBPF). The exploration of such a substructure is certainly not new. Here we extend the derivation of the RBPF with the inclusion of (19). The presentation given here is limited in scope and given to justify the correctness of the inclusion of a hard constraint on the non-linear states, not to provide a self-contained derivation of the RBPF. More in-depth treatment on the RBPF are in [20], [21],[22] from which the derivation given here draws heavily. Now, we can factorize the posterior density as
\[ p(X_n^k, x_k^l|Y^k, C_k) = p(x_k|X_n^k, Y^k, C_k)p(X_n^k|Y^k, C_k) \] (19)
\[ = p(x_k^l|X_n^k, Y^k)p(X_n^k|Y^k, C_k). \] (20)

Practically, we want to update the conditionally linear and Gaussian states, \( p(x_l^k|X_n^k, Y^k, C_k), \) with the Kalman Filter and the non-linear states with the particle filter. The goal is to get the predicted, updated density \( p(X_n^{k+1}, x_l^{k+1}|Y^{k+1}, C^{k+1}) \). To get the predicted density \( p(X_n^{k+1}, x_l^{k+1}|Y^{k}, C_k) \), first the non-linear states are predicted
\[ p(X_n^{k+1}|Y^k, C_k) = \int p(x_n^{k+1}, x_l^{k+1}|X_n^k, Y^k, x_l^k)p(x_l^k|Y^k, x_l^k)dx_l^k \] (21)
The solution to the integral in (21) (see [22]) is given by
\[ p(x_n^{k+1}|X_n^k, Y^k, C_k) = p(x_n^{k+1}|X_n^k, Y^k) = \mathcal{N}(x_n^{k+1}|f^n(x_n^k) + A^n(x_n^k)x_l^k, A^n(x_n^k)P_k|x_l^k(A^n(x_n^k))^T + Q^n_k). \] (22)

Since the predicted non-linear states \( X_n^{k+1} \) contain new information about \( x_l^k, \) the conditionally linear states are updated with the predicted non-linear states, that is
\[ p(x_l^k|X_n^{k+1}, Y^k) = \frac{p(x_l^{k+1}|X_n^{k+1}, Y^k)p(x_l^k|X_n^k, Y^k)dx_l^k}{p(x_l^{k+1}|X_n^{k+1}, Y^k)}. \] (23)
This is needed when we now predict the linear states
\[ p(x_l^{k+1}|X_n^{k+1}, Y^k) = \int p(x_l^{k+1}|X_n^{k+1}, x_l^k, Y^k)p(x_l^k|X_n^{k+1}, Y^k)dx_l^k. \] (24)
The next step is to update with the latest measurement $y_{k+1}$ and with the latest constraints $C_{k+1}$.

$$p(x_{k+1}^n|X_{k+1}^n, Y^{k+1}, C_{k+1}) = \frac{p(y_{k+1}|x_{k+1}^n, Y^k)p(x_{k+1}^n|Y^k, C_{k+1})}{p(y_{k+1}|Y^k, C_{k+1})}.$$  \hspace{1cm} (25)

The solution to the integral in (26) (see [22]) is given by

$$p(y_{k+1}|X_{k+1}^n, Y^{k+1}) = \mathcal{N}(y_{k+1}|h(x_{k+1}^n) + g(x_{k+1}^n)x_{t_i}^l, g(x_{k+1}^n)R_{k+1}g(x_{k+1}^n)^T + R_k).$$  \hspace{1cm} (27)

The linear states are now updated with the latest measurement and constraint

$$p(x_{t_i+1}^l|X_{k+1}^n, Y^{k+1}, C_{k+1}) = \frac{p(y_{k+1}|x_{t_i+1}^l, x_{t_i}^l, X_{t_i}^l, Y^k, C_{k+1})}{p(y_{k+1}|x_{t_i}^l, X_{t_i}^l, Y^k, C_{k+1})}.$$  \hspace{1cm} (28)

Here, the fact that the linear state $x_{t_i+1}^l$ is independent of the latest constraint $C_{k+1}$ when conditioned on the non-linear trajectory has been used. Now we have

$$p(x_{t_i+1}^l|X_{k+1}^n, Y^{k+1}, C_{k+1}) = p(x_{t_i+1}^l|X_{k+1}^n, Y^{k+1}, C_{k+1}).$$  \hspace{1cm} (29)

finishing the recursion. Since the focus here is on the filtering distribution, that is $p(x_{t_i+1}^l, x_{t_i}^l|Y^{k+1}, C_{k+1})$, and not of the joint posterior density $p(x_{t_i+1}^l, x_{t_i}^l|Y^{k+1}, C_{k+1})$, we can approximate the filtering distribution by dropping the history of states. Algorithm 1 shows the particle filter approximation, RBPF, incorporating hard constraints.

In informally we can motivate the fact that (28) stays Gaussian by noticing that the inclusion of the constraint $C_{k+1}$ only affects the non-linear states via the generalized likelihood $p(C_{k+1}|x_{t_i}^n)$. and not the conditionally linear states. More formally a proof by induction is needed, identical to the proof given in e.g. [22].

V. CREATING A ROBUST PROPOSAL DISTRIBUTION

One of the issues that we would like to target in this article is to design a filter that performs well is cases when there is a lot of “information gained”\(^2\) by adding the constraint. The effectiveness of the methods described in section III-D and section III-E can then often be low. An idea on how to create a more robust algorithm would be to somehow incorportate the knowledge of the constraint in the proposal distribution.

\(^2\) We can informally think of this as when the integral of the unconstrained posterior over the constrained area is small.

Algorithm 1 Constrained Rao-Blackwellized Particle Filter

1: for all $i = 1, \ldots, N$ do (Initialize)
2: $x_{0|i}^l = x_{0i}^l$, $P_{0|i}^l = P_0$
3: $x_{0|i}^n = p(x_{0i}^n)$
4: Set $k = 1$
5: for all $i = 1, \ldots, N$ do (Prediction)
6: $x_{n|i}^n = N(y_{k+1}^n|A^n(x_{k+1-1}^n)^T + Q_{k+1}^n, P_{k+1}^n)$
7: $N_{k+1} = N_{k+1}^n - N_{k+1}^n = (A^n(x_{k+1-1}^n)^T + Q_{k+1}^n)$
8: $A^n_{k+1} = A^n_{k+1} - Q_{k+1}^n - Q_{k+1}^n - Q_{k+1}^n$
9: $L_{k+1} = L_{k+1}^n - L_{k+1}^n - L_{k+1}^n$
10: $Q_{k+1}^n = Q_{k+1}^n - Q_{k+1}^n - Q_{k+1}^n$
11: $z_k = x_{k|i} - f(x_{k|i}^n) - f(x_{k|i}^n)$
12: $x_{k|i}^l = f(x_{k|i}^n) + x_{k|i}^n(x_{k|i}^n) + x_{k|i}^n(x_{k|i}^n)$
13: $L_{k+1}^l = L_{k+1}^l - L_{k+1}^l - L_{k+1}^l$
14: for all $i = 1, \ldots, N$ do (Update)
15: $w_{i}^l = p(x_{k|i}^l) = p(x_{k|i}^l|x_{k|i}^n, Y_{k+1}^n)p(x_{k|i}^n|x_{k|i}^n, Y_{k+1}^n)w_{k-1}^l$ where
16: $p(x_{k|i}^l|x_{k|i}^n, Y_{k+1}^n)$ is given by (22)
17: $p(y_{k+1}^n|x_{k|i}^n, Y_{k+1}^n)$ is given by (27)
18: $w_{i}^l = \sum_{i} w_{i}^l$
19: Re-sample
20: $\tilde{y}_{k+1}^l = h(x_{k|i}^l) + g(x_{k|i}^l)x_{k|i-1}^l$
21: $S_{k+1}^l = g(x_{k|i}^l)P_{k+1}^lg(x_{k|i}^l)^T + R_k$
22: $K_{k+1}^l = P_{k+1}^lS_{k+1}^l^{-1}$
23: $x_{k|i}^l = x_{k|i}^l + K_{k+1}^l(\tilde{y}_{k+1}^l - \tilde{y}_{k+1}^l)$
24: $P_{k+1}^l = P_{k+1}^l - K_{k+1}^lS_{k+1}^lK_{k+1}^l$.
25: Set $k = k + 1$ and go to Prediction

Any proposal distribution that contains the support of the posterior distribution is valid. What we want is a distribution that is (a) easy to sample from, (b) has its support bounded by the constraints, and (c) catches the important parts of the posterior distribution. The first requirement is met by having a uniform density. The process noise, given in (1), is Gaussian and has infinite support. To create a proposal distribution that incorporates the support, we approximate the support of the Gaussian to be bounded at three standard deviations, for example. To clarify, this means that in the particle filter, we approximate the support of the prior for particle $i$, $p(x_i^n|x_i^{n-1})$ to be bounded. The intersection of this approximate support of $p(x_i^n|x_i^{n-1})$ and the constrained set is calculated and a uniform sample is drawn from this area.\(^3\) Depending on the shape of the constrained set, there are different ways of finding the

\(^3\) Alternatively one may consider intersecting with an approximate support of the likelihood $p(y_i|x_i^n)$, or by intersecting with both the supports of $p(x_i^n|x_i^{n-1})$ and $p(y_i|x_i^n)$. 

Algorithm 2 Problem specific proposal distribution

1: for all $i = 1, \ldots, N$ do
2: Find approximate support $S_k$ of $p(x^k_i | x^k_{i-1})$
3: Calculate $D_k = S_k \cap C_k$
4: Calculate the volume $V(D_k)$ of $D_k$
5: Draw a uniform sample from $D_k$
6: Calculate importance weight $q(x_k | x_{k-1}, y_k, C_k) = 1/V(D_k)$

intersection between the two sets and how to draw samples from it. In section VI, we will present two possible ways of doing this. In Algorithm 2 pseudo-code for the creation of the proposal distribution is given.

VI. Simulation

In this section we will apply the idea of using a problem specific proposal distribution to two scenarios. The first is an on-road tracking scenario, then we will try a similar approach in a more demanding littoral tracking application. For both scenarios, the sensor is a radar and the measurements are modeled by (30). The target motion is modeled in the filters by (32). For both scenarios, we compare:

- A particle filter running the pseudo measurement approach presented in section III-E.
- A RBPF with a uniform proposal distribution over the allowed area.

The scenario specific parameters are given in Table I.

A. Observation Model

The target is observed by a radar, giving noisy observation of the range and bearing to the target, that is

$$y_k = h(x^r_k) + e_k,$$

where

$$h(x^r_k) = \left(\begin{array}{c} \frac{\|x^r_k - x^s\|}{\arctan(2x^a_k, y^a_k)} \\ \arctan(2x^a_k, y^a_k) \end{array}\right).$$

Here $x^s, y^a$ are coordinates in a sensor based coordinate system. The location of the radar for each scenario is given in Table I. The measurement noise $e_k$ is Gaussian

$$e_k \sim \mathcal{N}(0, R_k),$$

with covariance

$$R_k = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2_{az} \end{pmatrix}.$$

B. Dynamical Model

The target dynamics is modeled with a Constant Velocity (CV) model [23]. The model is linear and given by

$$x_{k+1} = Ax_k + v_k,$$

where

$$A = \begin{pmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, $T$ denotes the sampling time. The process noise $v_k$ is assumed white and Gaussian

$$v_k \sim \mathcal{N}(0, Q_k),$$

where

$$Q_k = \begin{pmatrix} \frac{a_x T^3}{3} & 0 & \frac{a_x T^2}{2} & 0 \\ 0 & \frac{a_x T^3}{3} & 0 & \frac{a_x T^2}{2} \\ \frac{a_x T^2}{2} & 0 & a_x T & 0 \\ 0 & \frac{a_x T^2}{2} & 0 & a_y T \end{pmatrix}.$$
the intersection will always be a square and a uniform sample is easily drawn. If the road instead has an angle relative to the constraint, the intersection between the box and the constraints will in general not be a square. Since the model of the support as a square is an approximation to begin with, we approximate the intersection as a square. This simply means that we increase the size of our approximate support of the prior.

Fig. 2. The dashed box models the approximate support for the prior. The intersection of the road and the box gives the region from where to sample.

Simulation results: To measure the robustness and efficiency of the two filters, an approximation of the Effective Number of Samples ($N_{eff}$) is used [24]. The approximation of $N_{eff}$ is calculated as

$$N_{eff} = \frac{1}{\sum_{i=1}^{N} (w^i_{k|k})^2},$$

(34)

and a filter that provides higher values of $N_{eff}$ is considered more robust since more particles will be used in the description of the posterior. In Fig. 3, the mean value of the $N_{eff}$, calculated over 200 Monte Carlo runs using 4000 particles is depicted for the different road widths. The results indicate that the pseudo-measurement approach works well when the width of the road is large\(^4\), while the efficiency of the approach decreases for a decreasing road width. The results for the road-proposal are the opposite. To confirm that the output of the two filters is consistent, the Kullback-Leibler Divergence (KLD) between the two particle clouds is estimated over 100 Monte Carlo runs. The KLD measures the closeness of two probability distributions. A K-Nearest Neighbor estimator of the KLD, adapted to the case of particle representations given in [25], was used here to estimate the closeness of the two posterior distributions. Ideally, the mean value of the KLD should go towards zero as the number of particles goes towards infinity, empirically proving the posterior distributions equal and the effect of our approximation as negligible. In Fig. 4, the mean and the variance of the KLD estimator is given.

Fig. 3. The mean number of efficient samples for five different road widths, calculated over 200 Monte Carlo runs per road width. The road widths are (20, 15, 10, 5, 2.5)m, shown in the same order with a decreasing level of darkness.

E. Simulation two - littoral tracking

To further develop the results in the previous section, we look at a more complicated scenario, and consider the case of littoral tracking. The main difference from the previous simulation is that the external knowledge of possible target locations is now in the form of a map. A portion of the map is depicted in Fig. 5 together with the simulated trajectory. The target dynamics are still modeled with (32) and sensor measurements are given by (30). The parameters used in the scenario are given in Table I. As done previously, two filters are compared: a particle filter using the pseudo measurement approach and an RBPF using a constraint specific proposal distribution. The development of this proposal distribution will be presented next.

\(^4\)Large in relation to size of the region where most of the mass of the prior is located
A map specific proposal distribution: The design of a proposal distribution according to section V is not as immediate as for the previous case. Here, the shape of the constraints makes it hard to do draw samples directly from the constrained area. To overcome this difficulty, a new representation of the area is needed. In Fig. 6, a portion of the map in Fig. 5 is given. However, the area is now described with a nearly uniform grid of triangles instead of polygons. Here we have used the routines given in [26] for the triangulation. This change of representation makes it easier to create the proposal distribution. As done previously, the intersection of the approximate support of the prior and the constraints has to be calculated. In this simulation, we have assumed an isotropic noise distribution and therefore model the approximate support as a circle with a radius of three standard deviations. To find the intersection between the map and the approximate support, all triangles that have a vertex located within the circle are taken. To draw a uniform number from this intersection, first, the total area of the triangles in range is calculated and a weight is assigned to the triangle according to its relative area. Next, one of the triangles in range is randomly picked according to its weight. Lastly, a uniform number is drawn from inside this triangle.

Simulation results: As in the previous example, the effective number of samples is used as a measure of robustness. In Fig. 7, the mean $N_{eff}$ calculated over 200 Monte Carlo runs, using 2000 particles is given. The results show that in the beginning of the trajectory, the pseudo measurement approach provides a higher $N_{eff}$. This is due to the fact that the target has not yet entered the narrow part of the channel, see Fig. 5 and there $p(x_k|x_{k-1}^*)$ is a more efficient proposal distribution. As long as the target is traveling in the channel, the new approach provides a higher $N_{eff}$, indicating that this choice of proposal distribution can increase the robustness of the filter.

VII. CONCLUSIONS

When incorporating hard constraints, the choice of proposal distribution is an important design issue that has not been given much attention in the literature. In this article, a new
proposal distribution adapted to the constraint was developed. The method works complementary to the pseudo measurement approach by achieving a higher degree of efficiency when the information gain given by adding the constraint increases. The conditions for choosing method is a topic for future analysis, but it is the opinion of the authors that by combining the pseudo measurement approach with the proposed method, hard constraints can be included both efficiently and robustly in the particle filter.

ACKNOWLEDGMENT

This work has been partly financially supported by Swedish and Dutch MoD’s, as a European Defence Agency Research & Technology project (Contract B-0898-GEM4-ERG). This support is gratefully acknowledged.

REFERENCES