Dynamic estimation of the discernment frame in belief function theory

Wafa Rekik, Sylvie Le Hégarat-Mascle, Roger Reynaud
Institute of Fundamental Electronics, 91405 Orsay, cedex, France
Email: {wafa.rekik, sylvie.le-hegarat, roger.reynaud}@u-psud.fr

Abdelaziz Kallel, Ahmed Ben Hamida
University of Sfax, Tunisia
Advanced Technologies for Medicine & Signals
Email: abdelaziz.kallel@isecs.rnu.tn, ahmed.benhamida@enis.rnu.tn

Abstract—Belief function theory is now widely used in decision systems because of its ability to model both the imprecision and the uncertainty. Now, the problem of the discernment frame estimation is even more critical as the whole set of handled hypotheses is the powerset of the discernment frame.

In this work, we propose to update and adjust the discernment frame in a sequential way as new sources provide new information pieces. Besides incompleteness, we assume that the current discernment frame may contain duplicated or fictitious hypotheses. We thus propose new updating mechanisms and we show on a practical application, namely the videosurveillance, how these mechanisms work.

I. INTRODUCTION

Although being the first and the most used theory for representing uncertainty, the probability theory may not be flexible enough when lack of knowledge occurs. Indeed, it mainly represents the uncertainty of the information, whereas in many cases the information is partial and/or imprecise. To model the imprecision (ambiguity) in addition to uncertainty, the evidence theory was first introduced by Dempster [1] and then expanded by Shafer [11]. It generalizes the probability theory by allowing to handle not only singleton hypotheses (as in the case of probability) but also compound hypotheses (i.e. sets or disjunctions of hypotheses).

The set of hypotheses is called the discernment frame and noted Ω. It is supposed to be exclusive, i.e. its elements are mutually disjoint, and Ω elements are the singleton hypotheses. Under assumption of a closed world, the exhaustive is also assumed. It means that Ω includes all possible hypotheses (or values) for the variable of interest. Smets proposed [12] to rather consider an open world. It allows to allocate some confidence represented by a non null mass value to the empty set Φ. Now, the interpretation of a non null mass on the empty set is ambiguous, since it can derived either from a conflict between the sources or from a modeling problem, such as with missing hypotheses (open world model). Then, [4] defined an extended open world where a specific hypothesis different from Φ represents any missing hypothesis.

More generally, several authors worked on the automatic estimation of the frame of discernment. [6], [8] consider a multi sensor fusion problem where sources are homogeneous. The sources being complementary, they have different discriminative capacities. In [6], each source is defined in a non specific frame of discernment that is a subset of the common frame including all hypotheses considered by any source. Two deconditioning methods are presented to redefine the sources on the common frame. The first one is based on the Principle of Minimum Information (introduced by Smets [12]). The second deconditioning is based on additional information about compatibility relationships between hypotheses discerned by a source and hypotheses that cannot be discerned. In [8], the discernment frame is automatically learned from the intersections between the hypotheses respectively distinguished by individual sources.

In [10], Schubert presents an algorithm to construct the most adequate frame of discernment from a set of basic belief functions associated with sources that can be heterogeneous. The common discernment frame should then be a cross product between discernment frames associated to one or several sources homogenous in itself and heterogeneous with each other. Now, according to [10], only the cross product whose elements are exclusive sets and mutually disjoint is a possible frame of discernment. In a second step, from each possible frame a set of tighter spaces, called abridgments, is constructed. An abridgment is obtained by reducing at least one element of the cross product to one of its non empty subsets. [10] proposed an adhoc measure, called the Frame Appropriateness, to evaluate the quality of an abridgment. This method was illustrated on a very simple example where three sources give information regarding the color, the speed and the color of a car, respectively.

Now, besides but related to the problem of the automatic estimation of the discernment frame, we propose to consider the problem of the dynamic estimation and updating of the discernment frame. Conversely to the previously mentioned works, here we assume that new information pieces occur (e.g. new sources will be taken into account in the fusion process) which may induce not only an updating of the discernment frame but also of the basic belief assignments. Let us give two examples of application where this problem is relevant. The first one is the classification: As data is acquired, the information becomes more complete and more accurate, so that new classes (subclasses of previous classes or true new classes) appear. Also, some non obvious links between classes are discovered so that some classes may fuse in one (for instance, at the XIX\textsuperscript{th} century, the naturalist Huxley demonstrated the deep kinship between reptiles and...
The second one is the object detection: As data is acquired, the information becomes more complete and more accurate, so that new objects appear as well as objects assumed different are recognized as different parts of a same object (partially hided), and some new objects are detected that were False Negatives according to previously considered source(s).

In this article, we take this second example as an applicative illustration (precisely video surveillance application) of the proposed methodology to revise discernment frame in the case of belief function framework. Our approach is more general than previously mentioned works in the sense that it handles three kinds of modification: the addition of a new element (of \( \Omega \), i.e. a new hypothesis), the removal of one \( \Omega \) element, the fusion of two existing elements.

The paper is organized as follows. Section II briefly recalls the operators of the belief functions we needed, and introduces the used notations. Section III specifies the problem. Section IV presents the proposed approach and the derived methodology. Section V shows some results and statistical performance. Section VI summarizes the key points of this study and gives some perspectives of our work.

II. Background

In this section, we present the tools used to develop our method. Our definitions and interpretations of belief functions are those developed by Smets in his transferable belief model [12].

A. Belief functions

We denote \( \Omega \) the frame of discernment, i.e. a set of mutually exclusive hypotheses that can be taken by the variable of interest. The family of belief functions includes three basic functions defined from \( 2^\Omega \) to \([0, 1]\): the mass function \( m \), the belief function \( bel \) and the plausibility function \( pl \). For any subset \( A \) of \( \Omega \), they represent the belief in \( A \) and in none of its subsets (mass values are given only to hypotheses for which there is a direct evidence), the minimal belief committed to \( A \) and the maximal belief that can be transferred to \( A \), respectively. \( m \) is such that the sum on subset masses of \( \Omega \) is equal to 1. Belief functions are related by one to one relations, so that they may be represented by a generic notation: the basic belief assignment \( bba \). For instance, \( bel \) and \( pl \) are deduced from \( m \) as follows:

\[
\forall A \in 2^\Omega, bel(A) = \sum_{B \in 2^\Omega \setminus \emptyset, A \subseteq B} m(B), \tag{1}
\]

\[
\forall A \in 2^\Omega, pl(A) = \sum_{B \in 2^\Omega \setminus A \neq \emptyset} m(B). \tag{2}
\]

The hypotheses for which \( m(A) > 0 \) are the focal elements of the bba \( m \). When \( \Omega \) is the only focal element, the bba is called the void bba. It models complete ignorance. Note that under the open world assumption, \( \emptyset \) may also be a focal element. When \( m(\emptyset) > 0 \), it is usually said that it represents the degree of conflict.

The link between uncertainty representation on \( 2^\Omega \) and uncertainty representation on \( \Omega \) as in probability theory is provided by the pignistic probability transform. It is defined by Smets [12] as follows:

\[
\forall H \in \Omega, BetP(H) = \sum_{A \in 2^\Omega \setminus \emptyset, H \in A} \frac{m(A)}{|A|(1 - m(\emptyset))}. \tag{3}
\]

B. Operators

1) Discounting: When a source is not totally reliable, the discounting operator allows to take into account its unreliability. Given a bba \( m \), the discounted bba \( \hat{m} \) was first defined by Shafer as follows:

\[
\forall A \in 2^\Omega, \hat{m}(A) = \alpha m(A), \tag{4}
\]

\[
\hat{m}(\emptyset) = \alpha m(\emptyset) + (1 - \alpha),
\]

where \( \alpha \) is a factor \( \alpha \in [0, 1] \) representing the bba unreliability. When \( \alpha = 0 \), the discounted bba is equal to the void bba.

The discounting defined by Eq. 4 is the same for all focal elements of initial bba \( m \). Now, since, in some cases, the reliability of the source depends on the considered hypothesis, [9] proposed a contextual discounting that allows to discount differently the different hypotheses.

2) Conditioning: When a subset \( C \) of \( \Omega \) is certain, the initial bba \( m \) may be conditioned so that all its focal elements are included in \( C \). Then, there cannot be a non null mass outside of \( C \) except \( \emptyset \).

\[
\forall A \in 2^\Omega, m(A|C) = \sum_{B \in 2^\Omega \setminus A \supseteq B \cap C} m(B). \tag{5}
\]

Deconditioning is the dual operation of conditioning.

3) Discernment frame manipulators: \( \Omega \) is the set of pairs where the first and the second components respectively belong to \( \Omega_1 \) and \( \Omega_2 \); \( \Omega = \{ (\omega_{1i}, \omega_{2j}), \forall i \in \{1, \ldots, |\Omega_1|\}, \forall j \in \{1, \ldots, |\Omega_2|\} \} \).

Being given a bba \( m^{\Omega_1} \) on \( \Omega_1 \), the vacuous extension allows to compute the least committed bba \( m^{\Omega_1, \Omega_2} \) on \( \Omega = \Omega_1 \times \Omega_2 \), as follows:

\[
\forall A \in 2^{\Omega_2}, m^{\Omega_1, \Omega_2}(A) = \sum_{B \in 2^\Omega_1 \setminus A \times \Omega_2} m^{\Omega_1}(B). \tag{6}
\]

The marginalization is the dual operation of vacuous extension. Let consider a bba \( m^{\Omega_1} \) defined on \( \Omega = \Omega_1 \times \Omega_2 \), \( m^{\Omega_1, \Omega_2} \) denotes the marginalization result on \( \Omega_1 \). It is such that:

\[
\forall A \in 2^{\Omega_2}, m^{\Omega_1, \Omega_2}(A) = \sum_{B \in 2^{\Omega_1} \setminus Proj^{\Omega_1}(B) = A} m^{\Omega_1}(B), \tag{7}
\]

where \( Proj^{\Omega_1}(B) \) denotes the projection on \( \Omega_1 \). Note that, usually, after marginalization, the initial bba \( m^{\Omega_1} \) cannot be recovered.

C. Combination rules

Several combination rules have been proposed either conjunctive (e.g. [2], [11], [12]), disjunctive (e.g. [2], [11]) or hybrid (e.g. [3], [14]). A major drawback for hybrid rules is their non associativity. Conjunctive rules are the mostly
used since they specify the information provided by each source (even if it is at the cost of an increase of the conflict, represented by \( m(\mathcal{G}) \)). In the following we only present the two most popular conjunctive rules.

1) Orthogonal sum: The orthogonal sum was proposed by Dempster [11] to combine independent pieces of evidence. It assumes a closed world, so that \( m(\mathcal{G}) = 0 \) and conflict when it appears is reallocated through a normalization step. Let two bbas \( m_1 \) and \( m_2 \) defined on the same frame of discernment \( \Omega \). The combined bba \( m_\oplus \) is given by:

\[
\forall C \in 2^\Omega \setminus \mathcal{G}, \quad m_\oplus(C) = \frac{1}{1 - K} \sum_{(A,B) \in 2^\Omega \times 2^\Omega} m_1(A) m_2(B),
\]

where:

\[
K = \sum_{(A,B) \in 2^\Omega \times 2^\Omega, A \cap B = \mathcal{G}} m_1(A) m_2(B).
\]

2) Smets’s conjunctive rule: In the case of an open world, Smets removed the normalization in Eq. 8. Given two bbas \( m_1 \) and \( m_2 \) defined on \( \Omega \), the conjunctive rule writes:

\[
\forall C \in 2^\Omega, m_\otimes(C) = \sum_{(A,B) \in 2^\Omega \times 2^\Omega, A \cap B = \mathcal{G}} m_1(A) m_2(B).
\]

III. PROBLEM DEFINITION

Our problem consists in automatic updating the discernment frame as some new sources are considered. Let assume a discernment frame to update, called \( \Theta = \{\theta_1,...\theta_M\} \). We use the name \( \Theta \) to specify the final discernment frame that is for a decision between \( M \) \( \theta_i \) hypotheses. Now, the questions that occur are about the relevance of \( \Theta \) hypotheses, e.g. is \( \theta_i \), \( i \in \{1,...M\} \), a hypothesis that should be not considered (and thus removed) or is \( \theta_j, j > M \), a new hypothesis that should be considered (and thus added).

In order to model the confidence in \( \Theta \) hypotheses, let us construct another intermediate discernment frame \( \Omega \). \( \Omega \) is constructed as a cross-product space whose elements (one dimension of the cross-product space) are as follows. Any \( \Theta \) hypothesis whose relevance (and then existence in \( \Theta \)) should be determined and is independent from the other hypotheses, is associated to an element of the cross-product \( \{o_j, \delta_j\} \). \( o_j \) corresponds to the relevance of considered \( \theta_i \), and \( \delta_j \) to the irrelevance case. Therefore, \( \Omega = \times_{j=1}^{N} \{o_j, \delta_j\} \).

Such a model using a cross-product space for non exclusive hypotheses (in our case several \( \theta_i \) may belong to \( \Theta \)) meets those proposed by Haenni [5] to overcome a classic modeling error [15]. Besides, it allows freeing from the crumbling of the mass between all the hypotheses even when all hypotheses are fully believable (due to the mass function constraint of sum equal to 1). For instance, assuming all hypotheses are equiprobable, if \( M=10 \), the mass allocated to each hypothesis cannot exceed 1/10. In the following of the paper, we firstly focus on updating \( \Omega \) since it drives \( \Theta \) updating process.

Classically assuming a current discernment frame \( \Omega_1 \) associated to bba \( m_1 \), and a new bba \( m_2 \) associated to a discernment frame \( \Omega_2 \), one aims at constructing the common discernment frame \( \Omega_{1,2} \) and redefining the bba \( m_1 \) and \( m_2 \) in this new discernment frame (for instance in order to combine them). \( \Omega_1 \) and \( \Omega_2 \) may be cross-product spaces. According to [10], there are two conditions to meet for a discernment frame \( \Omega \) being a cross-product \( \omega_1 \times \ldots \times \omega_N \). (although expressed differently, these conditions meet [10]’s ones):

1) the cross-product elements \( \omega_i \) are homogeneous sets of exclusive elements: \( \forall (H_j, H_k) \in \omega_i \times \omega_i, j \neq k \Rightarrow H_j \cap H_k = \mathcal{G} \),
2) the cross-product elements \( \omega_i \) are independent heterogeneous sets: a \( \Omega \) hypothesis is a \( N \)-tuple where elements take independent values.

Now, to construct the common discernment frame \( \Omega_{1,2} \), we have first to define the relationships between \( \Omega_1 \) hypotheses and \( \Omega_2 \) ones. When, as usually, \( \Omega_1 \) is assumed to be correct even if possibly incomplete, these relationships would lead to the recognition of some \( \Omega_2 \) hypotheses either as hypotheses already present in \( \Omega_1 \) or as ‘new’ hypotheses. However, in our case, we assume that \( \Omega_1 \) may be erroneous. In particular it may include two kinds of errors among \( \Omega_1 \) hypotheses: first, there may be some duplicates (i.e. some hypotheses actually the same are thought to be different) and, second, there may be some fictitious hypotheses (i.e. some hypotheses that actually do not match any realization). If the first ones could be identified based on the relationships between hypotheses, the identification of the second ones requires a decision process.

To illustrate these words, let us consider our application example. The final aim is to label every image pixel between the different objects and the background, thus \( \Theta = \{\theta_0, \theta_1,...\theta_M\} \), where \( \theta_0 \) represents the background and \( \theta_i, i \in \{1,...M\} \), the \( M \) objects present in the scene. Now, these objects are unknown and have to be determined. Each source \( S_i \) corresponds to a detection algorithm result. Let assume it detects \( \{o_j\}_{j=1}^{n_i} \), objects assumed different and true. Then, when a new source provides its own set of detected objects, four cases can occur: either an object is detected in only one of the two sets of objects, the current set and the new one, or it is detected in both; in this latter case, either the association is unique or it is multiple. The actions involved on the discernment frame \( \Theta \) are the following: when an object previously detected is no longer detected, it may be removed, when an object previously undetected is now detected, it may be added, when a multiple association is found, the concerned objects may be fused. Now, the decision of discernment frame modifications (removal, addition, fusion) should be carefully taken. Therefore, we use belief functions on cross-product space \( \Omega \). Note that \( \Omega \) may only deal with a subpart of \( \Theta \). For instance in the cited detection application, it does not deal with background hypothesis \( \theta_0 \) that is assumed always relevant.

Let us finally mention some difference between our approach and [7] also proposing a model of fusion at object level. In [7], the objective was to detect the true objects for tracking.
However, the authors deliberately ignore (partial) occlusion, object splitting phenomena, considering they are independent problems and their sources are good enough so that an object is always represented by a unique plot. In this study, we consider object undetection (occlusion) and/or fragmentation (partial occlusion) occurs as well as false detections (false alarms).

IV. PROPOSED MODEL

In this section, in order to be more practical, we use the vocabulary associated to our application. For generalization, simply change ‘object detection’ by ‘hypothesis distinction’.

As said in previous section, we handle two discernment frames. The cross-product space \( \Omega = \times_{j=1}^N \{ \Omega_j \} \) where \( \Omega_j \) denote respectively the presence of a \( j \)th object and the absence of the \( j \)th object, and the \( \Theta \) discernment frame that deduces. Then, as new sources provide information, both \( \Omega \) (and the associated bba) and \( \Theta \) are updated, either adding, merging or removing hypotheses.

A. Cross-product space \( \Omega \) updating

The common discernment frame shall meet the first and the second conditions recalled in Section III. So, the elements of the cross-product space (one dimensional spaces) have to be exclusive sets (condition 2) and independent. In particular there cannot be any duplicate between different elements of the cross-product.

Let \( S_1 \) and \( S_2 \) be two sources distinguishing respectively, \( n_1 \) objects \( \Omega_j \) and \( n_2 \) objects \( \Omega_k \) and so defined on their respective initial discernment frames \( \Omega = \times_{j=1}^{n_1} \{ \Omega_j \} \) and \( \Omega_2 = \times_{j=1}^{n_2} \{ \Omega_k \} \). The relationships between \( \Omega_j \) and \( \Omega_k \) should give the correspondences between objects. Ideally, all objects are detected by both sources, so that there is a bijection between \( \Omega_1 \) and \( \Omega_2 \). Practically, some objects may be detected by only one of the two sources. Let \( \equiv \) denotes a relationship, assumed transitive, between two elements of \( \Omega_1 \) and \( \Omega_2 \) meaning that they represent the same object. The common discernment frame \( \Omega \) is

\[
\Omega_{1,2} = \times_{j=\{1...n_1\}} \{ \Omega_j \} \times_{k=\{1...n_2\}} \{ \Omega_k \}
\]

According Eq. 11, \( \Omega_{1,2} \) is defined based on \( \Omega_1 \) adding the objects of \( \Omega_2 \) that were not already present in \( \Omega_1 \) and removing the objects of \( \Omega_1 \) that are duplicate according to \( S_2 \).

B. \( \Omega \) bbas updating

Having updated the discernment frame, the bba has also to be updated. For this, let us introduce the partition \( \Pi_{1,2} = \{ \pi_1, \ldots, \pi_{1,2} \} \) \((\nu_{1,2} = \text{the cardinality of } \Pi_{1,2})\) of the set of the \( n_1 \) detections from \( S_1 \) such that:

\[
\forall (j, k) \in \{1...n_1\}^2, \exists (i, l) \in \{1...\nu_{1,2}\}^2
\]

\[
\begin{cases}
\{ \Omega_j \in \pi_{i,1}, \Omega_k \in \pi_{i,l} \\
\Rightarrow o_j^{\prime} \equiv o_k^{\prime}
\end{cases}
\]

Thus \( \nu_{1,2} < n_1 \). The upperscript \( \equiv \) on \( \Pi_{1,2} \) and \( \nu_{1,2} \) recalls that the object relationships are given by source \( S_2 \).

Following previously notations, let us note \( m_{1,2} \) the bba \( m_{1} \) reallocated from \( \Omega_1 \) to \( \Omega_{1,2} \).

\[
m_{1,2}^{\Omega_{1,2}} = \bigotimes_{i \in \{1...\nu_{1,2}\}} \left( \bigotimes_{j \in \pi_{i,1}} m_{1} \{ o_j^{\prime}, o_j \} \right)^{\Omega_{1,2}}
\]

where \( \otimes \) denotes the operator which combines the beliefs associated to the \( S_1 \) detections that were thought to be associated with different objects and have been recognized as corresponding to the same object. Since this operator may depend on the application, it is not specified here. According to Eq. 13, the \( S_1 \) bba is marginalized over each element of the cross-product space \( \Omega_1 \), so that when duplicates are detected, the associated beliefs are combined. Then, the beliefs associated to each element of the partition \( \Pi_{1,2} \), i.e. different objects according to the current state of knowledge, are vacuously extended on \( \Omega_{1,2} \) (defined by Eq. 11). Finally, the \( \nu_{1,2} \) beliefs on \( \Omega_{1,2} \) are combined to get an unique belief associated to \( S_1 \) on the discernment frame common between \( S_1 \) and \( S_2 \). Note that when each element of the partition \( \Pi_{1,2} \) is of cardinality 1 (i.e. \( \nu_{1,2} = n_1 \), i.e. no detection have to be gathered), Eq. 13 boils down to a vacuous extension.

Having performed the same bba reallocation for \( m_{2} \), \( m_{1,2} \) and \( m_{2,3} \) may be combined to get \( m_{1,2,3} = m_{1,2} \circ m_{2,3} \).

Now the question that arises when considering three sources \( S_1, S_2 \) and \( S_3 \) is: In which case \( m_{1,2,3} = m_{1,2} \circ m_{2,3} \), or in other words, in which case the reallocation of intermediate combinations of sources is equal to the combination after reallocation of all individual sources? On the first hand,

\[
m_{1,2,3} = \bigotimes_{i \in \{1...\nu_{1,2}\}} \left( \bigotimes_{j \in \pi_{i,1,2}} m_{1,2} \{ o_j^{\prime}, o_j \} \right)^{\Omega_{1,2,3}}
\]

where \( \Pi_{1,2,3} \) is the partition (of cardinality \( \nu_{1,2,3} \)), given by \( S_3 \), on the objects detected using \( S_1 \) and \( S_2 \).

On the second hand,

\[
m_{1,2,3} = \left[ \bigotimes_{i \in \{1...\nu_{1,2}\}} \left( \bigotimes_{j \in \pi_{i,1}} m_{1} \{ o_j^{\prime}, o_j \} \right)^{\Omega_{1,2,3}} \right]
\]

\[
m_{2,3} = \left[ \bigotimes_{i \in \{1...\nu_{1,2}\}} \left( \bigotimes_{j \in \pi_{i,2}} m_{2} \{ o_j^{\prime}, o_j \} \right)^{\Omega_{1,2,3}} \right]
\]

Now, since the beliefs about the different objects are modelised on different elements of the cross-product space, the only beliefs that are actually combined are beliefs about the same object, either whether these different beliefs have been brought by different sources or whether these different beliefs have been brought by a same source (and recognized subsequently as corresponding to the same object). Besides, since the \( \equiv \) relationship is transitive (as in our application, see
further Eq. 21), no matter in which order the equivalences between detections are discovered. Then, \( m_{1,2}^{\Omega_{1,2,3}} = m_{1}^{\Omega_{1,2,3}} \ominus m_{2}^{\Omega_{1,2,3}} \) only if

\[
\forall (o_j, o_k) / o_j \equiv o_k, \\
\left[ m_{1}^{\Omega_{1,2,3}} (o_k) \otimes m_{2}^{\Omega_{1,2,3}} (o_k) \right] \ominus \left[ m_{1}^{\Omega_{1,2,3}} (o_j) \otimes m_{2}^{\Omega_{1,2,3}} (o_j) \right] \tag{17}
\]

where \( m_{1}^{\Omega_{1,2,3}} \) denotes the bba \( i \) marginalized on the cross-product element corresponding to the considered object, i.e. \( o_j \) or \( o_k \). Then, \( \ominus = \ominus \) ensures \( m_{1,2}^{\Omega_{1,2,3}} = m_{1}^{\Omega_{1,2,3}} \ominus m_{2}^{\Omega_{1,2,3}} \) and then the associativity of the discernment frame and bba updating.

C. Discernment frame \( \Theta \) updating

Finally the updated bbas are used to decide the set of actual objects. We propose two ways to take such a decision. The first one is a classical decision according to the maximum of pignistic probability (Eq. 3). Now, due to the form of the discernment frame, namely cross-product of two-element sets, the pignistic probability writes [13]

\[
\forall H \in \Omega / H = (H_1, ..., H_N), H_i \in \{ o_i, \bar{o}_i \}, \quad BetP (H) = \prod_{i=1}^{N} \text{BetP}(A_{1,2,3} (\bar{o}_i, o_j)) (H_i). \tag{18}
\]

Then, maximizing the pignistic probability boils down to choose, for every potential object, the hypothesis maximizing the marginal mass among \( \{ o_i, \bar{o}_i \} \).

The second way to decide the set of actual objects is to choose the abridgment [10] minimizing the conflict represented by the mass of \( \emptyset \). According to [10], when an abridgment is adopted, the beliefs are adjusted as for conditioning (Eq. 5). Thus,

\[
\forall H \in \Omega / H = (H_1, ..., H_N), H_i \in \{ o_i, \bar{o}_i \}, \quad m (\emptyset / H) = \sum_{A: A \cap H = \emptyset} m_{\emptyset}^{\Omega} (A) = \prod_{i=1}^{N} \text{BetP}(A_{1,2,3} (\bar{o}_i, o_j)) (H_i) - \sum_{i,j} \prod_{i=1,i \neq j}^{N} \text{BetP}(A_{1,2,3} (\bar{o}_i, o_j)) (H_i, H_j) + \sum_{i,j,k} \prod_{i=1,i \neq j \neq k}^{N} \text{BetP}(A_{1,2,3} (\bar{o}_i, o_j)) (H_i, H_j, H_k) \tag{19}
\]

Let us now consider an additional constraint on discernment frame cardinality: we assume that only \( M \) objects among the \( N \) potentially detected objects should be kept. The two criteria presented just bellow still occur, except that the maximization and the minimization are now performed not considering any hypothesis of \( \Omega \) but only the hypotheses of \( \Omega \) having \( M \) elements \( o_i \) and \( N - M \) elements \( \bar{o}_i \):

\[
\forall H \in \Omega / H = (H_1, ..., H_N), H_i \in \{ o_i, \bar{o}_i \}, \quad H \cap x_{j=1}^{N} o_j = M, |H \cap x_{j=1}^{N} o_j| = N - M. \text{ Finally note that } \Theta \text{ is directly deduced from } H. \text{ For instance in our application it is the set of the elements } o_i \in H \text{ and the background, i.e. } \theta_0.
\]

V. APPLICATION AND RESULTS

In this section, we aim at providing some results to evaluate the interest of the proposed approach. For this, let us specify the data provided by the different sources, the used relationships between hypothesis sets, and the way the monosource bba are defined.

A. Input data and object relationship

For the estimation of the actual set of objects, only sources providing information at object level are considered. For instance, considering videosurveillance application, we deal with change detection relative to a background image with a decision at window level (i.e. per block of connected pixels). Then, from binary image of changes, a connected-component labeling process provides the objects with different labels and so discernable.

In our application case, the relationship \( \equiv \) between the elements of monosource discernment frame meaning that they represent the same object is based on the spatial overlap between ‘plots’. Let us assume a first set of objects \( \{ o'_1, ..., o'_{n_1} \} \) (e.g. detected by \( S_1 \) or resulting of previously combined observations) and a \( S_2 \) second set of objects \( \{ o''_1, ..., o''_{n_2} \} \). We note \( A (o'_i) \) or \( A (o''_j) \) their spatial extension in the image domain. By construction, \( \forall (i, j) \in \{ 1...n_1 \}^2 \text{ s.t. } i \neq j, A (o'_i) \cap A (o''_j) = \emptyset \) and similarly for \( S_2 \) objects. Then, two ‘plots’ are recognized to be subparts of a same object (and thus would be gathered and re-labeled)

- either because they overlap:
  \[
  \forall (i, j) \in \{ 1...n_1 \} \times \{ 1...n_2 \}, \quad \frac{|A(o'_i) \cap A(o''_j)|}{\min \{ |A(o'_i)|, |A(o''_j)| \}} > \epsilon \Rightarrow o'_i \equiv o''_j, \tag{20}
  \]
  where \(|A|\) denotes the cardinality of \( A \) and \( \epsilon \) is a threshold parameter,
- or by transitivity (i.e. the ‘junction’ is made by one or several other plots):
  \[
  \forall (i, j) \in \{ 1...n_1 \}^2, \quad \exists k \in \{ 1...n_2 \} \text{ / } \begin{cases} o'_i \equiv o''_k \quad o''_k \equiv o'_j \end{cases} \Rightarrow o'_i \equiv o'_j. \tag{21}
  \]

To illustrate the common discernment frame construction, we take an example where the source \( S_1 \) detects four objects \( o'_1, o'_2, o'_3 \) and \( o'_4 \) so that \( \Omega_1 = x_{j=1}^{4} \{ o'_j, o'_j \} \) and the source \( S_2 \) detects two objects so that \( \Omega_2 = x_{j=1}^{2} \{ o''_j, o''_j \} \).
Eq. 20, we find \( o'_1 \approx o''_1, o'_2 \approx o''_2, o'_3 \approx o''_3 \), the common discernment frame will be \( \Omega_{1-2} = \times_{j=1}^{3} \{ \alpha'_j, \alpha''_j \} \) where \( \alpha_1 \approx \alpha'_1 \approx \alpha''_1, \alpha_2 \approx \alpha'_2 \approx \alpha''_2 \) and \( \alpha_3 \approx \alpha'_3 \approx \alpha''_3 \); in other words, \( \alpha_1 \) was fragmented in two subplots according to \( S_1 \) and \( \alpha_3 \) was misdetected by \( S_2 \).

B. bba definition on an initial discernment frame

The initial bba allocation \( m_i \) aims at formalizing the information regarding the true objects in the scene provided by \( S_i \). For this, we assume each object \( o_j \) detected by \( S_i \) is a source of information concerning its real existence in the scene. Then we define a bba \( m_{ij} \) on the exclusive and exhaustive object discernment frame \( \Omega_{ij} = \{ o_j, \bar{o}_j \} \).

Assuming the number of pixels in \( o_j \) is the only available information about that object (in our application), and the more important is this number, the more we are confident in the fact that \( o_j \) is a true detection, the mass \( m_{ij}(o_j) \) on \( o_j \) is defined based on this number noted \( |A(o_j)| \).

Then, all bbas \( m_{ij} \) are vacuously extended on \( \Omega_i \) and conjunctively combined to get the bba \( m_{i}^{\Omega_i} \):

\[
 m_{i}^{\Omega_i} = \bigotimes_{j=1}^{n_i} m_{ij}^{\Omega_{ij}} \cap \Omega_i,
\]

where \( n_i \) is the number of objects detected by \( S_i \) source generating as many bbas \( m_{ij} \).

For the following we assume that our sources are such that there are more false negatives than false positives. So, when an object \( o_j \) is detected with only few pixels, low mass values are assigned to both \( o_j \) and \( \bar{o}_j \) whereas an important mass is assigned to \( \Omega_{ij} \), since only few pixels are not informative about the existence or the non-existence of an object in the scene.

We now present two ways to define the bba \( m_{ij}^{\Omega_{ij}} \). In the second case, \( \Omega \) bbas updating (IV-B) ensures the associativity of \( \Theta \) estimation. In the first case, to ensure the associativity of \( \Theta \) estimation, the bbas should be recalculated as new \( \equiv \) relationships are discovered.

1) linear allocation: For this bba allocation, the bba \( m_{ij} \) depends linearly on the number \( |A(o_j)| \) of pixels in the object \( o_j \) as follows:

\[
\begin{align*}
 m_{ij}(o_j) &= \min_{1} \left( 1, \frac{|A(o_j)|}{\tau_a} \right), \\
 m_{ij}(\bar{o}_j) &= p \left( 1 - \min_{1} \left( 1, \frac{|A(o_j)|}{\tau_a} \right) \right), \\
 m_{ij}(\Omega_{ij}) &= (1 - p) \left( 1 - \min_{1} \left( 1, \frac{|A(o_j)|}{\tau_a} \right) \right),
\end{align*}
\]

where \( \tau_a \) is an a priori parameter representing the number from which the object is certain, \( p \approx 0.01 \) is a parameter to weighten the mass given to \( \Omega_{ij} \) relatively to the mass given to \( o_j \). \( m_{ij} \) is then discounted (Eq. 4) with a factor \( a \in [0, 1] \) to take into account the unreliability of the source and to get a non-dogmatic bba.

2) Non linear allocation: According to this approach, we assume that each pixel \( s \) of an object \( o_j \) is an information source. Then, we represent it through a bba \( m_s \) defined on the discernment frame \( \{ o_j, \bar{o}_j \} \).

\[
\begin{align*}
 m_s(o_j) &= x, \\
 m_s(\bar{o}_j) &= \epsilon (1 - x), \\
 m_s(\Omega_{ij}) &= (1 - \epsilon) (1 - x),
\end{align*}
\]

with \( \epsilon \) and \( x \) two a priori parameters. Then the bba \( m_{ij} \) about the existence of the object \( o_j \) according to source \( S_j \) (all pixels) is the result of the conjunctive combination of the bbas per pixel. It is achieved using the orthogonal sum (Eq. 8) in order not to introduce conflict due to the bba allocation process: \( m_{ij} = \bigoplus_{s \in A(o_j)} m_s \), i.e. in this allocation case we choose \( \otimes = \bigoplus \) as used in IV-B. Then, denoting \( a_j = |A(o_j)| \),

\[
\begin{align*}
 m_{ij}(o_j) &= \frac{[1 + \epsilon(x - 1)] a_j - (1 - \epsilon) a_j + (1 - x) a_j}{1 - \epsilon^a}, \\
 m_{ij}(\bar{o}_j) &= \frac{1 - (1 - \epsilon) a_j}{1 - \epsilon^a}, \\
 m_{ij}(\Omega_{ij}) &= \frac{(1 - \epsilon^a)(1 - x) a_j}{1 - \epsilon^a},
\end{align*}
\]

where \( K \) is the Dempster’s conflict (mass on the empty set before normalization):

\[
K = 1 - [1 + \epsilon(x - 1)] a_j - (1 - \epsilon) a_j [1 - (1 - \epsilon) a_j].
\]

The parameters \( \epsilon \) and \( x \) are estimated in an ad hoc way so that the mass on an object of size \( \tau_a \), i.e. almost certain, is equal to \( \eta \) (in our case \( \eta = 0.98 \)), i.e. solving the equation:

\[
(1 - \eta) [1 + \epsilon(x - 1)] a_j - (1 - \epsilon) a_j + \eta (1 - (1 - \epsilon) a_j) = 0.
\]

The size of the object of interest is given by the application (for instance, in our case, for an image of size \( 600 \times 256 \), we choose \( \tau_a = 30 \times 30 \) pixels). In our tests, varying \( \eta \) between 0.8 and 0.98, results are close (the impact of variations in the parameters is illustrated further on results).

Fig. 1 shows an example of the proposed method. The detections provided by three different sources are combined so that they are fused when they represent the same object, added when they represent a new object, and removed when they are not supported enough. Adding a fourth source, the discernment frame is updated so that we are more and more confident in the detected objects relatively to possible false alarms or non detections. On fusion results (Fig. 1d and 1f) each hypothesis of discernment frame \( \Theta \) is represented by a different color, whereas the source individual detections have been reported as edges, so that the false alarms detected appear as edges on white (background). We note that, in this toy example, when the fourth source is added some subparts of the green (or of the yellow) object are recognized as belonging to the same object, illustrating the fusing hypothesis process. We also note that using \( \Theta \) keeps all information and allows to eventually reintroduce some hypotheses previously removed from \( \Theta \), as for the car in the upper right corner of the image (coded in white in Fig. 1d, i.e. as a false alarm at this stage of the fusion, and in orange in Fig. 1f, i.e. as an object using the fourth source also).
C. Results

We now present some results obtained on simulated data. We use simulated data in order to be able to evaluate quantitatively the performance of the proposed method. Simulations are performed as follows. We simulate a ground truth image containing \(n_T\) ‘true’ objects, and a false alarm image containing \(n_T\) ‘false’ objects (note that such a simulation process allows the correlation of false alarms detected by the different sources). Each object is composed of a set of connected fragments. Other simulation parameters are the probability of non-detection, noted \(p_{nd}\) and the probability of false alarm, noted \(p_{fa}\). Then, for each source \(S_i\) the observation is simulated as follows: For each object \(o_j\), for each fragment \(o_{i,j}\) of \(o_j\), randomly drawn if it is detected knowing \(p_{nd}\) (uniform drawing of \(x\) in \([0,1]\), \(S_i\) contains \(o_{i,j}\) if \(x \geq p_{nd}\) for each false alarm \(o_j\), randomly drawn if it is present knowing \(p_{fa}\) (uniform drawing of \(x\) in \([0,1]\), \(S_i\) contains \(o_{i,j}\) if \(x < p_{fa}\). From the simulation of a set \(S\) of \(|S|\) sources (from ground truth and false alarm maps and \((p_{nd}, p_{fa})\) parameters), let \(C\) be the subset of \(S\) of the considered sources. For a given \(S\), \(C\) is varied by introducing the sources one per one from \(|C| = 2\) to \(|C| = |S|\).

For method quantitative performance estimation, we call ‘sample’ a triplet \((TP, FP, \text{FN})^{|C|}_{(p_{nd}, p_{fa})}\) representing the numbers of ‘true positive’, ‘false positive’ and ‘false negative’, i.e. the numbers of true objects well detected, of false alarms detected as objects and of true objects not detected. Note that when an object is fragmented, one fragment is labeled as a \(TP\) and the others as \(FP\). We focus on the dependency on the simulation parameters and on the number of considered sources. Hence samples are averaged for fixed values of \((p_{nd}, p_{fa}), |C|\).

Fig. 2 shows the obtained results either setting \(|C|\) and varying \((p_{nd}, p_{fa})\) or vice-versa. The x-axis is the ‘positive predictive value’ \((PPV = \frac{TP}{TP + FP})\) and the y-axis is the ‘sensitivity’ \((S_e = \frac{TP}{TP + FN})\). When \(|C|\) is fixed, the different curves represent the results corresponding to different values for \((p_{nd}, p_{fa})\), and the points on a given curve represent the result obtained for a given number of kept objects among all the potentially detected objects (parameter \(M\) introduced at the very end of Section IV-C). When this number \(M\) increases (from 1 to 12) in the case of Fig. 2, both the number of \(FN\) decreases and the number of \(TP\) (or \(FP\)) increases, then both \(S_e\) increases and \(PPV\) decreases as observed on the curves. Now, from Fig. 2 first column, we clearly see that the lower are \((p_{nd}, p_{fa})\) values, the closer to 1 are \((S_e, PPV)\) values, and from Fig. 2 second column, the higher is the source number \(|C|\), the closer to 1 are \((S_e, PPV)\) values. Comparing Fig. 2 first and second lines, we also note that the difference between curves are all the more significant that the monosource performance is low or the number of sources is low. Comparing Fig. 2 second and third lines, we note that the global conclusions are the same whatever the bbas allocation (among the two proposed), but that local results differ without one being significantly better. Comparing Fig. 2 third and fourth lines, same conclusions occur for \(\eta = 0.98\) and \(\eta = 0.8\).

Finally, we compare the four instantiations of our methodology, i.e. using linear or non linear bba allocation and using decision based on pignistic probability or empty set mass. Here, in order to have a scalar performance indicator, we consider the area delimited by the performance curve obtained varying \(M\) (cf. Fig. 2) and the x-axis and y-axis. Perfect results correspond to a value equal to 1, so that the closer to 1 the indicator is, the better is the performance achieved. Depending on the \((p_{nd}, p_{fa})\) values and the number of considered sources (that vary as for Fig. 2), the respective performance for the four instantiations of our methodology varies in \([0.8761, 1], [0.8497, 1], [0.875, 1]\) and \([0.8125, 1]\).

Besides the global performance of the methods, let us now focus on their difference. For this, let us introduce the following notations: \(LB\) stands for the method using linear bba allocation and maximum pignistic probability decision, \(NLB\) stands for the method using non linear bba allocation and maximum \(BetP\) decision, \(LA\) stands for the method using linear bba allocation and minimum \(m(\emptyset/H)\) abridgment decision and \(NLA\) stands for the method using non linear bba allocation and minimum \(m(\emptyset/H)\) abridgment decision. Then we compute all the differences between performance results (for given \((p_{nd}, p_{fa})\) and source number) obtained by any pair within \([LB, NLB, LA, NLA]\) methods.

Tab. I presents the statistics of these differences, in terms of mean value, \(L_1\) and \(L_{inf}\) norm. We note that these differences are very small, as it already appeared on Fig. 2.

VI. CONCLUSION

In this study, we focused on dynamic estimation and updating of the discernment frame. We assumed that the current discernment frame may be partially erroneous in addition
to be partially incomplete. Thus, the possible discernment frame modifications are the removal of some hypotheses, the addition of new hypotheses, and the fusion of several hypotheses. To robustify the decision of modifications, another discernment frame has been introduced that is a cross-product space whose elements correspond to the different hypotheses whose relevance is in question.

To illustrate our methodology, we consider a detection application. Assuming a videosurveillance system, images are acquired in order to detect events (e.g. intruders, called in a generic way ‘objects’) relatively to a background image. Estimating the discernment frame is equivalent to enumerate the different objects actually present in the scene. It is also a preliminary step to the image classification that will have to decide, for each pixel, which object or background it represents.

Future work will deal with more complete tests of the proposed methodology and the use of the estimated discernment frame for classification in an actual videosurveillance system.

REFERENCES