Optimal $H_\infty$ Filtering for Discrete-Time-Delayed Chaotic Systems via a Unified Model

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Abstract—This paper presents a unified model, consisting of a linear dynamic system and a bounded static nonlinear operator. Most discrete-time chaotic systems, such as chaotic neural networks, Chua’s circuits, and Hénon map etc, can be transformed into this unified model. Based on the $H_\infty$ performance analysis of the estimation error system between the unified model and its improved Luenberger-like filter using the linear matrix inequality (LMI) approach, the optimal $H_\infty$ filter are designed to estimate the states of discrete-time chaotic systems with external disturbance. The $H_\infty$ filter not only guarantees the asymptotic stability of the estimation error system, but also reduces the influence of noise on the estimation error. Two numerical examples are exploited to illustrate the effectiveness of the proposed filter design schemes.

Keywords—$H_\infty$ filtering; estimation error; discrete-time chaotic systems; time delay; neural networks.

I. INTRODUCTION

Chaotic system is a very complex dynamical nonlinear system and its response exhibits some specific features such as excessive sensitivity to initial conditions, broad Fourier transform spectrums, and irregular identities of the motion in phase plane [1]. The chaotic systems and relevant properties have found useful applications in many engineering areas such as secure communication, biological systems, power electronic devices and power quality, digital communication, chemical reaction analysis...[2]. During past two decades, research about control and synchronization of chaotic systems has gained increasing attention [1-12]. However, many existing approaches depend on the assumption that their states can be detected by sensors, which is not always the case in reality. Therefore, from a practical viewpoint, the state estimate problem for chaotic systems has been vitally important. On the other hand, some noises or disturbances always exist in real systems that may cause instability and poor performance. Therefore, the effect of the noises or disturbances must be also reduced in state estimation process for chaotic systems. In this regards, recently, many researchers adopted the $H_\infty$ filter to estimate systems’ states in order to guarantees the $L_2$ gain less than a prescribed level. It should be point out that numerous research results regarding synchronization problems for chaotic systems based on the observer techniques have been developed. For the observer-based synchronization, a slave system is designed such that its dynamics synchronizes that of the master system. From the viewpoint of control theory, the slave system is an observer of the master system, and the state of the master system can be estimated by the slave system. In this sense, the synchronization problem can be viewed as a state estimation one [13]. There are many results regarding $H_\infty$ synchronization problems for chaotic systems [14-19].

However, to our best knowledge, the above aforementioned methods and many other existing synchronization methods are only applied to the continuous-time chaotic systems. It seems not much (if any) study on the $H_\infty$ synchronization or filtering for discrete-time chaotic systems with external disturbance. It is well known that discrete-time systems play a very important role in digital signal analysis and processing. Especially, the discrete-time chaotic systems are more suitable to realize time-series prediction, adaptive tracking, and secure communication, etc. Therefore, here we will combine the $H_\infty$ control concept and Lyapunov (or Lyapunov-Krasovskii) stability theory to investigate the optimal $H_\infty$ filtering problem for a class of discrete-time chaotic systems with external disturbances. We are inspired by the standard neural network model (SNNM) in [19-21] and put forward a unified model, which is the interconnection of a linear dynamic system and a bounded static nonlinear operator. Most discrete-time chaotic systems with time delays, such as chaotic neural networks, Chua’s circuits, and Hénon map, etc, can be transformed into this unified model to be $H_\infty$ filter designed in a unified way. The contributions of this paper include: (i) A unified model is presented to describe different kinds of chaotic systems. (ii) The improved Luenberger-like filter is less conservative than general filters. (iii) The $H_\infty$ filter will reduce the effect of noise or disturbance with bounded energy on the estimation accuracy.

Notation: The superscript “T” stands for matrix transposition. $l_2(0, \infty)$ is the space of square integrable vectors. $\Re^n$ denotes n dimensional Euclidean space, and $\Re^{m \times n}$ is the set of all $m \times n$ real matrices. $I$ denotes identity matrix of
appropriate order. * denotes the symmetric parts, diag \{ \ldots \} stands for a block-diagonal matrix. The notations \( X>Y \) and \( X \geq Y \), where \( X \) and \( Y \) are matrices of same dimensions, mean that the matrix \( X>Y \) is positive definite and positive semidefinite, respectively. If \( X \in \mathbb{R}^p \) and \( Y \in \mathbb{R}^q \), \( C(X; Y) \) denotes the space of all continuous functions mapping \( \mathbb{R}^p \rightarrow \mathbb{R}^q \).

II. PROBLEM FORMULATION

The unified model we suggested consists of a linear dynamic system and a bounded static nonlinear operator:

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + A_x x(k-\tau) + B_x \phi(\zeta(k)) + B_w w(k), \\
\dot{\zeta}(k) &= C_x x(k) + C_{\phi} \phi(\zeta(k)) + D_x w(k),
\end{align*}
\]

with the initial condition function \( x(k)=\mathcal{A}x(k), \) \( \forall k \in [-\tau, 0] \), where \( x(k) \in \mathbb{R}^n \) is the system state, \( A \in \mathbb{R}^{n \times n}, A_x \in \mathbb{R}^{n \times n}, B_x \in \mathbb{R}^{n \times d}, B_w \in \mathbb{R}^{n \times m}, C_x \in \mathbb{R}^{l \times n}, C_{\phi} \in \mathbb{R}^{l \times n}, D_x \in \mathbb{R}^{l \times d}, \) and \( D_w \in \mathbb{R}^{l \times m} \) are the corresponding state-space matrices, \( \zeta \in \mathbb{R}^m \) is the input of nonlinear function \( \phi, \zeta \in \mathbb{R}^m \) is nonlinear function satisfying \( \phi(0)=0 \). \( w(k) \in \mathbb{R}^m \) is the disturbance input which belongs to \( l_2(0, \infty), L \in \mathbb{N} \) is the number of nonlinear functions, \( \mathbb{R} \) is the time delay, \( \mathcal{A} \) is the given function on \( [-\tau, 0] \).

Remark 1: While \( w(k)=0 \), the nonlinear model (1) unifies linear systems, several well-known discrete-time intelligent systems including dynamic neural networks or fuzzy models with or without time delays, discrete-time Chua’s circuits, and Henon map. Ref. [19]-[23] illustrate that some intelligent systems are special examples of (1). We will provide some other examples in Section IV.

Suppose that the measurement of the system (1) by the sensor is of the form:

\[
y(k) = Cx(k) + Dw(k),
\]

where \( y(k) \in \mathbb{R}^l \) is the measurement output, \( u(k) \in \mathbb{R}^d \) is the measurement noise that belongs to \( l_2(0, \infty) \). \( C \in \mathbb{R}^{l \times n} \) and \( D \in \mathbb{R}^{l \times m} \) are known constant matrices.

The signal to be estimated is the combination of the system state described as follows:

\[
z(k) = C_x x(k),
\]

where \( z(k) \in \mathbb{R}^r \) is the non-measurable signal to be estimated, \( C_x \in \mathbb{R}^{r \times n} \) is a constant matrix. We construct the following improved Luenberger-like filter for \( z(k) \):

\[
\begin{align*}
\hat{x}(k+1) &= Ax(k) + A_x \hat{x}(k-\tau) + B_x \phi(\hat{\zeta}(k)) \\
&+ K_x[y(k) - C\hat{x}(k)] + K_x[y(k-\tau) - C\hat{x}(k-\tau)], \\
\hat{\zeta}(k) &= C_x \hat{x}(k) + C_{\phi} \hat{\zeta}(k-\tau) + D_x \phi(\hat{\zeta}(k)) \\
&+ K_x[y(k) - C\hat{x}(k)] + K_x[y(k-\tau) - C\hat{x}(k-\tau)], \\
\hat{z}(k) &= C_x \hat{x}(k),
\end{align*}
\]

where \( K_x \in \mathbb{R}^{n \times l}, K_x \in \mathbb{R}^{l \times l}, K_x \in \mathbb{R}^{l \times l} \) and \( K_x \in \mathbb{R}^{l \times l} \) are the filter gains to be determined to meet certain performance criteria. \( \hat{x}(k) \) and \( \hat{z}(k) \) denote the estimates of \( x(k) \) and \( z(k) \), respectively. By defining the error vector \( e(k) = x(k) - \hat{x}(k) \), we get the following dynamic equations that \( e(k) \) satisfies.

\[
\begin{align*}
e(k+1) &= \overline{A}e(k) + \overline{A}_x e(k-\tau) \\
&+ B_w f(\hat{\zeta}(k)) + B_w w(k) - K_x \mathcal{D}v(k), \\
\tilde{\zeta}(k) &= \zeta(k) - \hat{\zeta}(k) = \mathcal{C}_q e(k) + \mathcal{C}_{\phi q} e(k-\tau) \\
&+ D_x f(\hat{\zeta}(k)) + D_{\phi w} w(k) - \mathcal{K}_x \mathcal{D}v(k), \\
\tilde{z}(k) &= z(k) - \hat{z}(k) = C_x e(k),
\end{align*}
\]

where \( \overline{A} = A - K_x C \), \( \overline{A}_x = A_x - K_x C \), \( \mathcal{C}_q = C_q - K_x C \), \( \mathcal{C}_{\phi q} = C_{\phi q} - K_x C \), \( \mathcal{K}_x = [K, K] \), \( \mathcal{K}_x = [K, K] \), \( \mathcal{D} = \text{diag}\{D, D\} \) \( f(\hat{\zeta}(k)) = \phi(\hat{\zeta}(k)) - \phi(\hat{\zeta}(k)) \), \( v(k) = [\phi^T(k) \psi^T(k)]^T \), and \( \tilde{z}(k) \) is the estimation error.

If there exists a positive scalar \( \gamma \) such that

\[
J(\mathcal{W}(k), v(k)) = \sum_{t=0}^{\infty} \tilde{z}^T(k) \tilde{z}(k) - \gamma^2 [w^T(k) w(k) + \psi^T(k) \psi(k)] < 0,
\]

for any nonzero \( w(k) \in l_2(0, \infty), v(k) \in l_2(0, \infty), \) with the initial state \( x(k)=\mathcal{A}x(k), \forall k \in [-\tau, 0] \), and system (5) is asymptotically stable when \( w(k)=0, v(k)=0 \), then the \( L_2 \) gain of system (5) does not exceed \( \gamma \); that is, system (4) is an \( H_\infty \) filter for \( z(k) \). If we find a minimal positive \( \gamma \) to satisfy the above conditions, system (4) is an optimal \( H_\infty \) filter for \( z(k) \).

III. MAIN RESULTS

In this paper, we assume that the nonlinear functions in (1) are monotonically non-decreasing and globally Lipschitz. That is, there exist a positive scalar \( \gamma \) such that

\[
0 \leq \frac{\phi(\alpha) - \phi(\beta)}{\alpha - \beta} \leq \gamma, \forall \alpha, \beta \in \mathbb{R}, i=1, \ldots, L.
\]

Theorem 1: If there exist symmetric positive definite matrices \( P \) and \( \Gamma_i \), diagonal positive semi-definite matrix \( \Sigma \), matrices \( K_1, K_2, K_3, \) and \( K_4 \), and a positive scalar \( \gamma \) that satisfy

\[
0 \leq \frac{f(\hat{\zeta}(k))}{\gamma(k)} \leq \gamma, \text{ i.e., } f(\hat{\zeta}(k)) \cdot [f(\hat{\zeta}(k)) - u_i \hat{\zeta}(k)] \leq 0, \\
u_i > q \geq 0, i=1, \ldots, L.
\]
\[
M = \begin{bmatrix}
\bar{A}^T P A - P & \bar{A}^T P A_d - \Gamma & \bar{A}^T P B_p + \tilde{C}_w^T \Sigma U \\
\bar{A}^T P A_d - \Gamma & \bar{A}^T P A_d - \Gamma & \bar{A}^T P B_p + \tilde{C}_w^T \Sigma U \\
\bar{A}^T P A_d - \Gamma & \bar{A}^T P A_d - \Gamma & \left(B_p^T P B_p + D_p^T \Sigma U \right) + U \Sigma D_p - 2 \Sigma \\
\end{bmatrix}
\]

where \( U = \text{diag}\{u_1, u_2, \ldots, u_L\} \), then system (5) with \( w(\cdot) = 0 \) and \( v(\cdot) = 0 \) is globally asymptotically stable and the upper bound on the \( L_2 \) gain of the system (5) is finite and can be obtained by minimizing \( \gamma \) with respect to \( \gamma, \Gamma, \Sigma, K_1, K_2, K_3, \) and \( K_4 \), i.e.,

\[
\begin{align*}
\text{minimize} & \quad \gamma, \\
\text{subject to} & \quad (9), P > 0, \Gamma > 0, \Sigma \geq 0. \\
\end{align*}
\]

**Proof:** First, consider system (5) with \( w(\cdot) = 0 \) and \( v(\cdot) = 0 \); that is,

\[
\begin{align*}
\dot{e}(k) &= \bar{A} e(k) + \bar{A}_d e(k-\tau) + B_p f(e(k)), \\
\dot{z}(k) &= \bar{C}_w e(k) + \bar{C}_w e(k-\tau) + D_p f(e(k)), \\
\end{align*}
\]

Since \( e(0) = 0 \) and \( \bar{z}(0) = 0 \) are solutions to (11), there exists at least one equilibrium located at the origin, i.e. \( e_0 = 0, \bar{z}(0) = 0 \).

For system (11), we adopt the following Lyapunov-Krasovskii functional:

\[
V(e(k), \bar{z}(k)) = e^T(k) P e(k) + \sum_{i=1}^{L} e^T(i+k) \Gamma e(i+k),
\]

where \( P > 0 \) and \( \Gamma > 0 \). Thus, \( \forall e(k) \neq 0, \forall \bar{z}(k) \neq 0, V(e(k), \bar{z}(k)) > 0 \) and \( V(e(k), \bar{z}(k)) = 0 \) iff \( e(k) = 0 \) and \( \bar{z}(k) = 0 \). From the sector condition (8), we have

\[
f_i(e(k), \bar{z}(k)) - f_i(e(k), \bar{z}(k)) - u_i \bar{z}_i(k) \leq 0,
\]

where \( \varepsilon_i \geq 0 \) \( (i=1, \ldots, L) \). From the equality (13), we have

\[
2 \varepsilon_i f_i(e(k), \bar{z}(k)) - 2 \varepsilon_i u_i f_i(e(k), \bar{z}(k)) \bar{z}_i(k) \leq 0,
\]

The difference of \( V(e(k), \bar{z}(k)) \) along the solution to (11) is

\[
\Delta V(e(k), \bar{z}(k)) = V(e(k+1), \bar{z}(k+1)) - V(e(k), \bar{z}(k)) \\
\leq \left[ \bar{A} e(k) + \bar{A}_d e(k-\tau) + B_p f(e(k)) \right] ^T P \left[ \bar{A} e(k) + \bar{A}_d e(k-\tau) + B_p f(e(k)) \right] \\
- e^T(k) P e(k) + e^T(k) \left( \Gamma e(k) - \tau e^T(k) \Gamma e(k) \right) G \\
- 2 \sum_{i=1}^{L} \varepsilon_i u_i f_i(e(k), \bar{z}(k)) \bar{z}_i(k) \\
\]

where

\[
G = \begin{bmatrix}
\bar{A}^T P A - P + \Gamma & \bar{A}^T P A_d - \Gamma & \bar{A}^T P B_p + \tilde{C}_w^T \Sigma U \\
\bar{A}^T P A_d - \Gamma & \bar{A}^T P A_d - \Gamma & \left(B_p^T P B_p + D_p^T \Sigma U \right) + U \Sigma D_p - 2 \Sigma \\
\end{bmatrix}
\]

Next, for system (5) under zero initial condition, \( J \) in Eq. (6) is equivalent to

\[
J(w(k), v(k)) = \sum_{i=0}^{\infty} \{ \bar{z}^T(i) \bar{z}(k) - \gamma^2 [w^T(i)w(k) + v^T(i)v(k)] \} \\
= \sum_{i=0}^{\infty} \{ \bar{z}^T(i) \bar{z}(k) - \gamma^2 [w^T(i)w(k) + v^T(i)v(k)] + \Delta V(e(k), \bar{z}(k)) \} \\
- \{ V(e(\infty), \bar{z}(\infty)) - V(e(0), \bar{z}(0)) \} \\
\leq \sum_{i=0}^{\infty} \{ \bar{z}^T(i) \bar{z}(k) - \gamma^2 [w^T(i)w(k) + v^T(i)v(k)] + \Delta V(e(k), \bar{z}(k)) \} \\
\leq \sum_{i=0}^{\infty} \{ e^T(i) C_w^T C_w e(k) - \gamma^2 [w^T(i)w(k) + v^T(i)v(k)] \} \\
+ \left[ e^T(k) e(k) - \tau e^T(k) \gamma \tilde{C}_w(k) \right] G \\
+ \left[ e^T(k) e(k) - \tau e^T(k) \gamma \tilde{C}_w(k) \right] T \\
+ [B_w w(k) - \bar{K}_w Dv(k)] ^T P [B_w w(k) - \bar{K}_w Dv(k)] \\
+ [B_w w(k) - \bar{K}_w Dv(k)] ^T P [\bar{A} e(k) + \bar{A}_d e(k-\tau) + B_p f(e(k))] \\
+ [\bar{A} e(k) + \bar{A}_d e(k-\tau) + B_p f(e(k))] ^T P [B_w w(k) - \bar{K}_w Dv(k)] \\
+ w^T(k) D_q^T \Sigma f(\tilde{z}(k)) + f^T(\tilde{z}(k)) \Sigma D_q w(k) \\
- v^T(k) D_q \tilde{K}_q^T \Sigma f(\tilde{z}(k)) - f^T(\tilde{z}(k)) \Sigma U \Sigma D_q w(k) \} \\
\]
\[
\begin{align*}
&= \sum_{i=0}^{\infty} \begin{bmatrix} e^T(k) & e^T(k-\tau) & f^T(\xi(k)) & w^T(k) & v^T(k) \end{bmatrix} \bar{M} \\
&\times \begin{bmatrix} e^T(k) & e^T(k-\tau) & f^T(\xi(k)) & w^T(k) & v^T(k) \end{bmatrix}^T.
\end{align*}
\]

Since \( M < 0 \) in inequality (16), \( J(w(k), v(k)) < 0 \) for any \([e^T(k) e^T(k-\tau) f^T(\xi(k)) w^T(k) v^T(k)]^T \neq 0\), \( w(k) \in \mathcal{L}_2[0, \infty) \), and \( v(k) \in \mathcal{L}_2[0, \infty) \). With the well-known Schur complement [24], \( M < 0 \) is equivalent to

\[
\begin{bmatrix}
\bar{A}^T P \bar{A} - P + \Gamma & \bar{A}^T P \bar{B}_p + \bar{C}^T_1 \Sigma U \\
\bar{A}_p^T P \bar{A}_d - \Gamma & \bar{A}_p^T P \bar{B}_p + \bar{C}^T_2 \Sigma U \\
* & * \\
B_p^T P \bar{B}_p + \Sigma D_{w2} & 0 \\
B_w^T P \bar{B}_w - \gamma^2 I & D^T \bar{K}_{12}^T P \bar{K}_{12} - \gamma^2 I \\
* & -\gamma^2 I
\end{bmatrix} < 0. \tag{17}
\]

Since \( G \) in Eqn. (15) is the principal minor of the left-hand side of Inequality (17), we have \( G < 0 \). So system (5) with \( w(k)=0 \) and \( v(k)=0 \), i.e. system (11), is globally asymptotically stable.

We expect that \( \gamma \) reaches its minimum so that system (5) can reject the external disturbance as much as possible. It requires the solution of the eigenvalue problem (EVP) in the inequality (10), which is a convex optimization problem that can be solved using the MATLAB LMI Control Toolbox [25]. This completes the proof.

Based on Theorem 1, we can obtain the following theorem to design an optimal \( H_\infty \) filter for the nonlinear system (1) with the observation system (2).

**Theorem 2**: There exists an optimal \( H_\infty \) filter (4) such that system (5) is globally asymptotically stable when \( w(k)=0 \), \( v(k)=0 \), and the upper bound of the \( L_2 \) gain of system (5) is minimal, provided that there exist symmetric positive definite matrices \( P \) and \( \Gamma \); a diagonal positive semi-definite matrix \( \Sigma \); matrices \( S_1, S_2, S_3, S_4 \), and \( S_5 \); and a positive scalar \( \gamma \) that satisfy the following EVP:

\[
\begin{align*}
\text{minimize} & \quad \gamma, \tag{18}
\end{align*}
\]

subject to

\[
\begin{bmatrix}
-P & PA - S_1 C & PA_d - S_3 C \\
* & -P + \Gamma + C^T_1 \Sigma_z & 0 \\
* & * & -\Gamma \\
* & * & * \\
* & * & *
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
P B_p & P B_w & -\bar{S}_{12} \bar{D} \\
C^T_1 \Sigma U - C^T S_3 & 0 & 0 \\
C^T_2 \Sigma U - C^T S_4 & 0 & 0 \\
D^T \Sigma U + U \Sigma D_p - 2 \Sigma & U \Sigma D_{w2} & -\bar{S}_{14} \bar{D} \\
* & -\gamma^2 I & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0, \tag{19}
\]

where \( \bar{S}_{12} = [S_1, S_2] \), \( \bar{S}_{14} = [S_3, S_4] \). Furthermore, the parameters of the desired \( H_\infty \) filter (4) can be determined by:

\[
\begin{align*}
K_1 &= P^{-1} S_1, \\
K_2 &= P^{-1} S_2, \\
K_3 &= (U \Sigma)^{-1} S_3, \\
K_4 &= (U \Sigma)^{-1} S_4.
\end{align*} \tag{20}
\]

**Proof**: With the well-known Schur complement [24], the equality (9) in Theorem 1 is equivalent to

\[
\begin{bmatrix}
-P & PA \\
* & -P + \Gamma + C^T_1 \Sigma_z \\
* & * \\
* & * \\
* & *
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
P B_p & P B_w & -\bar{S}_{12} \bar{D} \\
C^T_1 \Sigma U & 0 & 0 \\
C^T_2 \Sigma U & 0 & 0 \\
D^T \Sigma U + U \Sigma D_p - 2 \Sigma & U \Sigma D_{w2} & -U \Sigma K_{14} \bar{D} \\
* & -\gamma^2 I & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0. \tag{21}
\]

Defining \( S_1 = PK_1, S_2 = PK_2, S_3 = U \Sigma K_3, \) and \( S_4 = U \Sigma K_4 \) in (21), we can obtain Theorem 2. We complete the proof.
in which $x(k) = [x_1(k) \ x_2(k)]^T$, with the initial condition $[x_1(0) \ x_2(0)]^T = [0.4 \ 0.6]^T$ for $-1 \leq k \leq 0$, and $\Delta t = 1$,

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & -0.1 \\ -5 & 4.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix}.$$

Figure 1 shows the chaotic behavior of the system (22). Taking process noise $w(k)$ into account, which belongs to $l_2[0, \infty)$, the system (22) can be reformulated as follows:

$$x(k+1) = (1+\Delta t A_0)x(k) + \Delta t A_1 \tanh(x(k)) + \Delta t B \tanh(x(k-1/\Delta t)) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k). \quad (23)$$

We convert the delayed Hopfield neural network (23) into expression (1), where $A=1+\Delta t A_0$, $A_d = 0_{2x2}$, $B_p = [\Delta t A_1 \ \Delta t B]$, $B_u = [1 \ 1]$, $C_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_{ql} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $D_p = 0_{4x4}$,

$D_{qw} = 0_{4x1}$, $U = I_{4x4}$, $\phi(\xi(k)) = \tanh(x(k), i = 1, 2)$, $\phi(\xi(k)) = \tanh(x_1(k-1/\Delta t), i = 2)$. If $x_2(k)$ is estimated, we have

$$z(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k). \quad (24)$$

We employ the following sensor to observe the system’s information:

$$y(k) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k). \quad (25)$$

The process noise $w(k) \in l_2[0, \infty)$, and the measurement noise $v(k) \in l_2[0, \infty)$ are defined as follows.

$$w(k) = r_1 \sin(k) e^{-k}, \quad (26)$$

$$v(k) = \frac{r_2}{k}, \quad (27)$$

where $r_1$ and $r_2$ are random numbers taken from a uniform distribution over [0, 1]. Based on the sensor (25), we obtain the following gains of optimal $H_\infty$ filter (4) by virtue of Theorem 2.

$$K_1 = \begin{bmatrix} -0.0005 & 0.0002 \\ 0.0051 & -0.0022 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0030 & -0.0013 \\ -0.0074 & 0.0033 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 1.8625 & -0.9405 \\ -2.8670 & 1.9425 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0.0016 & -0.0007 \\ 0.0042 & -0.0019 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} 0.0008 & 0.0004 \\ -2.9932 & 1.9970 \end{bmatrix}.$$
The optimal upper bound on the $L_2$ gain of the system (23) with the sensor (25) is 10.1286. Simulation result is presented in Figure 2 which shows the state $x_2(k)$ and its estimate $\hat{x}_2(k)$ (i.e. $z(k)$).

Su et al. [28] have provided the synchronization methods of the chaotic system (30) where $\Delta t \leq 1/7$, however, the synchronization error diverged when $\Delta t > 1/7$. Here our approaches can deal with the state estimation problem of the chaotic system (22) in which $\Delta t \leq 1$, that is, Theorem 2 is less conservative than theorems in [28].

Example 2: Consider the following delayed chaotic system [29, 30]:

$$
\begin{align*}
    x_1(k+1) &= 0.5x_1(k) + T \tanh(x_1(k-2)), \\
    x_2(k+1) &= 0.5x_2(k) + T \sin(x_1(k-2)), \\
    x_3(k+1) &= 0.5x_3(k) + T \tanh(x_2(k-2)).
\end{align*}
$$

(28)

while $T \leq 1.52$, system (28) will exhibit chaotic behavior. For example, for $T = 1.94$ and initial conditions:

$$
\begin{align*}
    x(-2) &= [-2.86584, 0.438859, 1.73719]^T, \\
    x(-1) &= [-3.12197, 1.91191, 0.620975]^T, \\
    x(0) &= [-3.24297, 2.1437, 1.88233]^T,
\end{align*}
$$

the trajectory of the system (28) is shown in Figure 3. If we introduce the process noise $v(k) \in L_2[0, \infty)$ and $w(k) = \sin(2k) \times \exp(-2k)$ into the system (28), it becomes:

$$
\begin{align*}
    x_1(k+1) &= 0.5x_1(k) + T \tanh(x_1(k-2)) + w(k), \\
    x_2(k+1) &= 0.5x_2(k) + T \sin(x_1(k-2)) + w(k), \\
    x_3(k+1) &= 0.5x_3(k) + T \tanh(x_2(k-2)) + w(k).
\end{align*}
$$

(30)

We transform the chaotic system (30) into expression (1), where $A = \text{diag} \{0.5, 0.5, 0.5\}$, $A_d = [-T \ 0 \ 0], B_p = \text{diag} \{T, 0\}$, $B_v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $C_q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $D_{pq} = 0_{3\times1}$, $D_{qw} = 0_{3\times1}$, $U = \text{diag} \{1, 2, 1\}$, $\phi(t) = \tanh(x_3(k-2))$, $\phi(t) = x_1(k-2) + \sin(x_1(k-2))$, and $\phi(t) = \tanh(x_2(k-2))$. If $x_3(k)$ is estimated, we have

$$
    z(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k).
$$

(31)

We employ the following sensor to observe the system’s information:

$$
    y(k) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k).
$$

(32)

The measurement noise $v(k) \in L_2[0, \infty)$ is defined in Eqn. (27). According to Theorem 2, solving the EVP (18)-(19), we obtain the gains of desired $H_\infty$ filter as follows:

![Figure 3. Trajectory of chaotic system (28) with $T=1.94$ and initial conditions (29) (5000 iterations have been plotted).](image)

![Figure 4. State $x_3(k)$ and its estimate $\hat{x}_3(k)$ .](image)

![Figure 5. The estimation error $\hat{z}(k) = x_3(k) - \hat{x}_3(k)$ .](image)
When the filter (24) with the above $K_1$, $K_2$, $K_3$, and $K_4$ is used to estimate the state $x_1(k)$ of chaotic system (30), the state $x_1(k)$ and its estimate $\hat{x}_1(k)$ are shown in Figure 4, and estimation error $\tilde{z}(k)$ is shown in Figure 5. It can be seen that the effect of the noise $w(k)$ and $v(k)$ on the estimation error $\tilde{z}(k)$ can be restricted in the lowest level.

Although Ref. [30] provided synchronization methods of chaotic system (28), the influence of the noise or disturbance on synchronization controller hasn’t been considered. Besides, the gains of synchronization impulses were obtained by trial and error in Ref. [30]. Our methods not only reduce the effect of the noise or disturbance, but also solve the gains of the filters by the MATLAB LMI Control Toolbox [25].

V. CONCLUSION

In this paper, we have proposed $H_\infty$ filter design method for a class of delayed chaotic systems. We have presented a unified model to describe these chaotic systems. By employing the Lyapunov functional method combined with the $H_\infty$ control concept, an optimal $H_\infty$ filter has been designed to estimate the states of this unified model such that the upper bound on the $L_2$ gain is minimized. As most discrete-time chaotic systems with time delays can be transformed into this unified model, $H_\infty$ filter design for these systems can be done in a unified way. The resulting design equations are a set of LMIs which can be solved by the MATLAB LMI Control Toolbox [25]. Here, it is worth pointing out that there are no unified ways about how to convert the chaotic systems into the unified model (1), but generally state-transformation is applied.

ACKNOWLEDGMENT

This work was supported in part by the National Natural Science Foundation of China under Grants 61174142, 61071061 and 60874050, the Zhejiang Provincial Natural Science Foundation of China under Grants R1100234 and Z1090423, the Program for New Century Excellent Talents (NCET) in University under Grant NCET-10-0692, the Research Project of Zhejiang Provincial Education Department under Grant Z200909334, the Fundamental Research Funds for the Central Universities under Grants 111QNA4036 and 2009QNA4012, the ASFC under Grant 20102076002, and the SRFDP under Grant 20100121110055. This work was also supported by the “151 Talent Project” of Zhejiang Province.

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