A New Method of Multiple Attribute Decision Making Based on Possibility Degree

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Abstract—Multiple attribute decision-making theory under uncertainty is an important part of decision making theory. In this paper, by using optimal plan, the optimization weight model is established. Then, based on formulas of possibility degree, a new attribute decision making approach is proposed to study multi-criteria decision making problems under uncertainty in which criteria values are interval numbers and the information about alternative weights are known or uncertain completely. Finally, two examples are used to illustrate the feasibility and effectiveness of the presented method.

Keywords—multiple attribute decision making; Vague sets; possibility degree; preference

I. INTRODUCTION

Multiple attribute decision making has found wide application in society, economic, management, military and hard technology, and specifically plays an important role in fields of investment decision, project evaluation, economic benefit evaluation, staff appraisal and so on. Many real multiple attribute decision analysis (MADA) problems are characterized with both quantitative and qualitative attributes. So far many mature methods have been proposed to solve multiple attribute decision making problems[1-13], but most of them need to know the exact attribute values of alternatives from the first. And, most of these attributes are qualitative and could only be properly assessed using human judgments, which are subjective in nature and are inevitably associated with uncertainties caused due to the human beings inability to provide complete judgments, or the lack of information, or the vagueness of the meanings about attributes and their assessments. From this, it can be seen that the research about multiple attribute decision making under risk problem has great theoretical and realistic significance. Now, the research aiming at multiple attribute decision making with continuous random variable under risk are relatively less.

Literature [8] studies the risky multiple attribute decision making with continuous random variable on finite interval problems, and uses closeness degree with ideal solution to propose a random multiple attribute decision making method with incomplete weight information. Literature [9] presents TOPSIS rank method to solve risky multiple attribute decision making with weight information known and attribute values in the form of continuous random variables. Literature [10] discusses multiple Criteria decision making problem with weight information incomplete and Criteria values in the form of normally distributed random variables, and then develops a multiple criteria decision making method based on WC-OWA operator. Literature [11] places emphasis on risky multiple attribute decision making with attribute values in the form of continuous random variable on finite interval, and develops a new approach to handle this kind of problem.

Based on the additive multi-attribute value model, this paper analyzes the structure of the weight set and proposes a new method to determine the weight set while keeping the ranking orders on alternatives according to linear programming theory. Given a set of ordering on alternatives, the proposed method can also tell if the attribute weights are feasible or not. Thus it provides the necessary and sufficient conditions for decision makers to adjust weights while still keeping the ranking orders. This paper also lists the type of MADM decision models where the proposed conditions are applicable. Examples are used to illustrate the application of the proposed method. Section II of this paper introduces the multi-attribute value model. Section III introduces the standardization methods of decision matrix. Section IV presents the weight set for satisfying many preference orders of alternatives simultaneously. Section V puts forward the decision steps based on possibility degree. In Section VI, two examples are examined. Section VII summarizes the research outcomes and discusses the future work.

II. DESCRIPTION OF INTERVAL MADM

Definition 1: Let R be the real number domain. A bounded interval \([x^L, x^U]\) is called the interval number, which is expressed in \(\tilde{x}\), where \(x^L, x^U \in \mathbb{R}\), and \(x^L < x^U\).

This definition of interval number is used throughout the paper. Also, the following notations are used to represent a MADM problem with interval numbers:

\(S = \{S_1, S_2, \cdots, S_n\}\) : a discrete set of n possible alternatives.

\(Q = \{Q_1, Q_2, \cdots, Q_h\}\) : a set of h attributes. The attributes are additively independent. Without loss of generality, all attributes are assumed to be for maximization.

\(\omega = (\omega_1, \omega_2, \cdots, \omega_h)\) : the vector of attribute weights in the form of interval numbers, where \(\sum_{j=1}^{n} \omega_j = 1\), \(j = 1, \cdots, n\).

\(A = [a_{ij}]_{h \times n}\) : the decision matrix in the form of interval numbers, where \(a_{ij} = [a_{ij}^L, a_{ij}^U]\) denotes the consequence of
alternative $S_i$, with respect to attribute $Q_j$, $i=1,\ldots,h$, $j=1,\ldots,n$. Suppose $A$ is normalized, i.e. $a_{ij}^{L}, a_{ij}^{U} \in [0,1]$, $\forall i,j$.

The decision-maker’s objective is to choose $N(<n)$ most preferred alternatives or the most preferred alternatives $S^*$ from the set $S^* \subseteq S$.

Let $\omega = [\omega_1, \omega_2, \ldots, \omega_n]$ be the vector of attribute weights, where $\omega_j (j=1,\ldots,n)$ are the variables with unknown values.

Using the simple additive weighting method [12], the overall value (represented by the interval number $d_j = [d_j^{L}, d_j^{U}]$) of alternative $S_i$ can be expressed as

$$d_j^{L} = \sum_{j=1}^{n} d_j^{L} \omega_j^i, i=1,\ldots,m$$

$$d_j^{U} = \sum_{j=1}^{n} d_j^{U} \omega_j^i, i=1,\ldots,m$$

where $\omega_j^i (j=1,\ldots,n)$ and $\omega_j^i (j=1,\ldots,n)$ are the optimal solutions to the following pair of linear programming models, respectively,

$$\begin{align*}
\min d_j^{L} &= \sum_{j=1}^{n} d_j^{L} \omega_j \\
\text{s.t. } \omega_j^L &\leq \omega_j \leq \omega_j^U, j=1,\ldots,n \\
\sum_{j=1}^{n} \omega_j &= 1
\end{align*}$$

$$\begin{align*}
\max d_j^{U} &= \sum_{j=1}^{n} d_j^{U} \omega_j \\
\text{s.t. } \omega_j^L &\leq \omega_j \leq \omega_j^U, j=1,\ldots,n \\
\sum_{j=1}^{n} \omega_j &= 1
\end{align*}$$

It can be seen from (1) and (2) that to obtain the overall values of the alternatives (i.e. $d_1, d_2, \ldots, d_m$), $2m$ linear programming models must be solved, and the attribute weight vectors used may be different. The Inner Interval Prioritization Technique with Interval-To-Point Synthesis [13] is used to transform the resulting intervals into points to compare the alternatives [14].

For two alternatives, if there is an intersection between the evaluated intervals of the overall values of them, one is superior to the other by a possibility owing to the use of different sets of attribute weights. Transformation of the intervals of the overall values of the alternatives into crisp values (points) results in a loss of information of superiority possibility.

III. STANDARDIZATION METHODS OF DECISION MATRIX

Fetch $x = [x^L, x^U] = \{ x \mid x^L \leq x \leq x^U \}$, and $x$ is an interval number. Specially, if $x^L = x^U$, then $x$ is a real number. Now we give the calculating rules on interval number as follows:

Suppose that there are two interval numbers: $v = [v_1, v_2]$, $a_1 < a_2$, $b_1 < b_2$, and $k \geq 0$, then we can obtain:

$$(1) \quad a + b = [a_1 + b_1, \min(a_2 + b_2)]$$
$$(2) \quad a - b = [a_2 - b_1, \max(a_1 - b_1)]$$
$$(3) \quad ka = [ka_1, ka_2]$$

Because these intervals can be considered as a new specific faintness numbers, so methods and technique of fuzzy compositor are studied. When in the course of intervals MADA, which is mainly to standardize $A = [a_{ij}]_{hn}$ with intervals to $R = [r_{ij}]_{hn}$ by some given method and how to solve compositor problems of educating objects from intervals decision matrix.

Let $A = [a_{ij}]_{hn}$ be an attribute decision matrix, where $a_{ij} = [a_{ij}^{L}, a_{ij}^{U}]$ is the alternative $Q_j$ with respect to attribute $S_i$, $i \in H, j \in N$.

In general, there are benefit attribute values and cost attribute values in FMADM, and the different attribute values may exist in different dimensions. In order to measure all attribute values in dimensionless units and facilitate inter-attribute comparisons, each attribute value $a_{ij}$ in the matrix $A = [a_{ij}]_{hn}$ has to be normalized into a corresponding element $r_{ij}$ in the matrix $R = [r_{ij}]_{hn}$ by using Eq.(5) and Eq.(6), where $r_{ij} = [r_{ij}^{L}, r_{ij}^{U}], i \in H, j \in N$.

For benefit attribute $S_i$

$$r_{ij} = a_{ij} / \sum_{j=1}^{n} a_{ij}, i \in H, j \in N$$

For cost attribute $S_i$

$$r_{ij} = (1 / a_{ij}) / \sum_{j=1}^{n} (1 / a_{ij}), i \in H, j \in N$$

Using the operational laws of fuzzy numbers, Eq.(5) and Eq.(6) can be rewritten as:

For benefit attribute $S_i$

$$r_{ij} = a_{ij} / \sum_{j=1}^{n} a_{ij}, i \in H, j \in N$$

For cost attribute $S_i$

$$r_{ij} = (1 / a_{ij}) / \sum_{j=1}^{n} (1 / a_{ij}), i \in H, j \in N$$
\[
\begin{align*}
\begin{cases}
    r_{ij}^L &= a_{ij}^L / \sqrt{\sum_{i=1}^{n} (a_{ij}^L)^2}, \\
    r_{ij}^U &= a_{ij}^U / \sqrt{\sum_{i=1}^{n} (a_{ij}^U)^2},
\end{cases} \\
\text{For cost attribute } S_i
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    r_{ij}^L &= (1/a_{ij}^U) / \sqrt{\sum_{i=1}^{n} (1/a_{ij}^L)^2}, \\
    r_{ij}^U &= (1/a_{ij}^L) / \sqrt{\sum_{i=1}^{n} (1/a_{ij}^U)^2},
\end{cases} \\
\text{in the following sections, the superiority possibilities between alternatives are calculated from different perspectives. Accordingly, ranking methods for the alternatives are discussed.}
\end{align*}
\]

IV. NEW METHOD

A. A New Method based on possibility degree

After normalizing the elements of the interval number decision-making matrix, there is no difference between “benefit type” and “cost type” index. The greater the index value, the better. So when constructing the optimal scheme, we choose the scheme that have greatest interval number value in the normalized interval number decision-making matrix. The value of the interval number is the average number of the up limit and low limit of the interval number. So we can construct the optimal scheme of the multi-attribute decision-making problems as follows:

**Definition 2:** Assumed that

\[
\begin{align*}
    r_{ij}^L &= \max \{ r_{ij}^L / i = 1,2,\cdots,h \} \\
    r_{ij}^U &= \max \{ r_{ij}^U / i = 1,2,\cdots,h \}, \quad j = 1,2,\cdots,n
\end{align*}
\]

Its decision-making value is

\[
S^+ = \{ [n_{1}^L, n_{1}^U], [n_{2}^L, n_{2}^U], \cdots, [n_{n}^L, n_{n}^U] \}
\]

is the most optimal scheme.

When the sample set of every index is determined, \(S^+\) only changes with the normalized matrix \(R = [r_{ij}]_{h \times n}\).

Different \(S^+\) reflects different data structure characteristics. So, by solving minimization problem of the normalized matrix \(R = [r_{ij}]_{h \times n}\), the optimal variable can be determined, namely:

\[
\begin{align*}
\min d(\omega) &= \frac{h}{\sum_{i=1}^{n} \left[(r_{ij}^L - r_{ij}^L)^2 + (r_{ij}^U - r_{ij}^U)^2\right] \omega_i^2} \\
\text{s.t.} \quad \frac{h}{\sum_{i=1}^{n} \omega_i} = 1, \omega_i \geq 0, i \in H
\end{align*}
\]

This is a nonlinear optimization problem with \(\omega = (\omega_1, \cdots, \omega_{h})\) as optimization variable.

Then the weighted projection pursuit method is integrating \(n\) dimensional data into one dimension \(z_j\) which is in the projection direction of weighted optimum solution \(\omega = (\omega_1, \cdots, \omega_{h})\), namely the direction of \(\omega = (\omega_1, \cdots, \omega_{h})\), so

\[
z_j = \frac{h}{\sum_{i=1}^{n} r_{ij} \omega_i}
\]

B. Decision based on intervals

In order to compare and rank the alternatives, the evaluated intervals associated with the alternatives should be compared.

For simplicity, two interval numbers \(s_1 = [l_1, v_1]\) and \(s_2 = [l_2, v_2]\) denote the evaluation intervals of the overall values of alternatives \(s_1\) and \(s_2\), respectively. Without loss of generality, suppose that \(v_1 \geq v_2\). Thus, for the position relationship between \(s_1\) and \(s_2\).

To determine the relationship between \(s_1\) and \(s_2\) clearly, the following definitions are given.

**Definition 3:** Let two interval numbers be \(s_1 = [l_1, v_1]\) and \(s_2 = [l_2, v_2]\). If \(l_1 < l_2\) and \(v_1 = v_2\), we say that \(s_1\) is equal to \(s_2\), denoted as \(s_1 = s_2\).

**Definition 4:** Let two interval numbers be \(s_1 = [l_1, v_1]\) and \(s_2 = [l_2, v_2]\). If \(l_1 > v_2\), we say that \(s_1\) is larger than \(s_2\), denoted as \(s_1 \succ s_2\).

**Definition 5:** Let two interval numbers be \(s_1 = [l_1, v_1]\) and \(s_2 = [l_2, v_2]\). If one of the two end coordinates of one interval number (e.g. \(s_1\)) is larger than at least one of those of the other interval number (e.g. \(s_2\)), we say that \(s_1\) is superior to \(s_2\), denoted as \(s_1 \succ s_2\). If the median of \(s_1\) is larger than that of \(s_2\), we say that the superiority of \(s_1\) is preferable to that of \(s_2\).

**Definition 6:** Let two interval numbers \(s_1 = [l_1, v_1]\) and \(s_2 = [l_2, v_2]\) be the evaluated intervals of the overall values of alternatives \(s_1\) and \(s_2\), respectively. If there is an intercrossing part in them, the possibility of \(s_1 \succ s_2\) is

\[
p(s_1 \succ s_2) = \frac{\max \{ 0, (v_1 - l_1 + v_2 - l_2) - \max(0, v_2 - l_1) \} + v_1 + v_2 - l_1 - l_2}{v_1 + v_2 - l_1 - l_2}
\]

where the following three conditions are satisfied:

1. \(0 \leq p(s_1 \succ s_2) \leq 1\);
(2) \( p(s_1 \geq s_2) + p(s_2 \geq s_1) = 1 \) and \( p(s_1 \geq s_1) = \frac{1}{2} \);

(3) \( p(s_1 \geq s_2) \geq \frac{1}{2} \) only if \( v_1 + t_1 \geq v_2 + t_2 \), that is, \( v_1 - t_2 \geq v_2 - t_1 \);

(4) \( p(s_1 \geq s_2) \geq \frac{1}{2} \) and \( p(s_2 \geq s_3) \geq \frac{1}{2} \), then \( p(s_1 \geq s_3) \geq \frac{1}{2} \).

Proof:

Let’s prove that \( p(s_1 \geq s_2) \in [0, 1] \). Because \( 0 \leq 1 - v_1 + t_1 \leq 1 \), \( 0 \leq 1 - v_2 + t_2 \leq 1 \), so \( \max(0, v_2 - t_1) = 0 \) or \( \max(0, v_2 - t_1) = v_1 - t_2 \), thus \( 0 \leq p(s_1 \geq s_2) \leq 1 \).

\[ p(s_1 \geq s_2) + p(s_2 \geq s_1) = \max \{0, (v_1 - t_1 + v_2 - t_2) - \max(0, v_2 - t_1)\} + \max \{0, (v_1 - t_1 + v_2 - t_2) - \max(0, v_1 - t_2)\} \]

(1) If \( v_2 - t_1 \geq 0 \) and \( v_1 - t_2 \geq 0 \), then \( p(s_1 \geq s_2) = \frac{v_1 - t_2 - t_1}{v_1 - t_2} = 1 \);  

(2) If \( v_2 - t_1 > 0 \) and \( v_1 - t_2 < 0 \), or \( |v_2 - t_1| > |v_1 - t_2| \) for \( 0 \leq 1 - v_1 + t_1 \leq 1 \), \( 0 \leq 1 - v_2 + t_2 \leq 1 \), thus \( p(s_1 \geq s_2) + p(s_2 \geq s_1) = \frac{0 + v_1 - v_2 - t_1 + t_2}{v_1 - v_2 - t_1} = 1 \);

(3) If \( v_2 - t_1 < 0 \) and \( v_1 - t_2 > 0 \), then \( p(s_1 \geq s_2) + p(s_2 \geq s_1) = \frac{v_1 - v_2 - t_1 - t_2 + 0}{v_1 - v_2 - t_1 - t_2} = 1 \) for \( |v_2 - t_1| < |v_1 - t_2| \);  

(4) Specially, if \( s_1 = s_2 \), then \( v_1 = v_2 \), thus \max(0, v_2 - t_1) = v_1 - t_1 \) and \( p(s_1 \geq s_1) = \frac{v_1 - t_1}{2(v_1 - t_1)} = \frac{1}{2} \).

We first prove the sufficient condition. It is easy to check that \( v_1 + t_1 \geq v_2 + t_2 \), that is, \( v_1 - t_2 \geq v_2 - t_1 \).

(1) If \( v_2 - t_1 \leq 0 \), then \( p(s_1 \geq s_2) = \frac{\max \{0, v_1 - t_1 + v_2 - t_2\}}{v_1 - v_2 - t_1 - t_2} = 1 - \frac{1}{2} \);  

(2) If \( v_2 - t_1 > 0 \), then \( p(s_1 \geq s_2) = \frac{\max \{0, v_1 - t_1\}}{v_1 - v_2 - t_1 - t_2} = \frac{v_1 - t_2}{\frac{v_1 - v_2 - t_1 - t_2}{2}} \geq \frac{1}{2} \).

Secondly, we prove the necessary condition by contradiction. Let’s suppose \( v_1 - t_2 \geq v_2 - t_1 \).

(1) If \( v_2 - t_1 > 0 \), then \( p(s_1 \geq s_2) = \frac{\max \{0, v_1 - t_2\}}{v_1 - v_2 - t_1 - t_2} \leq \frac{1}{2} \), so it is in conflict with \( p(s_1 \geq s_2) \geq \frac{1}{2} \);

(2) If \( v_2 - t_1 < 0 \), then \( p(s_1 \geq s_2) = \frac{\max \{0, v_1 - t_2\}}{v_1 - v_2 - t_1 - t_2} = 1 - \frac{1}{2} \), so it is in conflict with \( p(s_1 \geq s_2) \geq \frac{1}{2} \).

According to definition 6, if \( p(s_1 \geq s_2) \geq \frac{1}{2} \), then \( v_1 + t_1 \geq v_2 + t_2 \); if \( p(s_2 \geq s_3) \geq \frac{1}{2} \), then \( v_2 + t_2 \geq v_3 + t_3 \).

So when \( v_1 + t_1 \geq v_3 + t_3 \), \( p(s_1 \geq s_3) \geq \frac{1}{2} \).

Definition 7: Suppose \( z = (z_1, z_2, \cdots, z_n) \) be a vector with \( z_j = [z^L_j, z^U_j] \), the possibility of \( z_i \succ z_m \) is defined

\[
p_{im}(z_i \succ z_m) = \frac{\max \{0, (z^U_i - z^L_i + z^L_m - z^U_m) - \max(0, z^L_m - z^L_i)\}}{z^L_i + z^L_m - z^L_i - z^L_m}
\]

According to definitions 7, a superiority possibility matrix \( P \) of pairwise comparisons on the evaluated intervals of the alternatives is set up as follows:

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}
\]

According to matrix \( P \), we can obtain the ranking result of the alternatives with superiority possibilities. A vector \( v = (v_1, v_2, \cdots, v_n) \), is defined as

\[
v_i = \frac{\sum_{l=1}^{n} P_{il}}{\sum_{i=1}^{n} \sum_{l=1}^{n} P_{il}} \quad l = 1, \cdots, n
\]

will help in ranking the alternatives. That is, according to the magnitude of \( v = (v_1, v_2, \cdots, v_n) \), together with the superiority possibilities in \( P \), the alternatives could be ranked.

V. METHOD OF FUZZY INTERVALS DAMA

The approach of fuzzy intervals MADA based on possibility degree is as follows:

The number decision making matrix of the scheme set \( S = (s_1, s_2, \cdots, s_n) \), and the evaluation index set \( A = \{a_{ij}\}_{ij\in n} \) of the multi-attribute decision-making problem, calculate the interval number decision making matrix of the scheme set \( S = (s_1, s_2, \cdots, s_n) \) on the index:


$$A = \begin{bmatrix}
[a_{11}, a_{11}], [a_{12}, a_{12}], & \cdots, & [a_{1n}, a_{1n}] \\
[a_{21}, a_{21}], [a_{22}, a_{22}], & \cdots, & [a_{2n}, a_{2n}] \\
\vdots & \vdots & \vdots \\
[a_{n1}, a_{n1}], [a_{n2}, a_{n2}], & \cdots, & [a_{nn}, a_{nn}]
\end{bmatrix}$$

Normalize the decision making matrix $A = [a_{ij}]_{m \times n}$, and get the normalized interval number decision making matrix:

$$R = \begin{bmatrix}
[r_{11}, r_{11}], [r_{12}, r_{12}], & \cdots, & [r_{1n}, r_{1n}] \\
[r_{21}, r_{21}], [r_{22}, r_{22}], & \cdots, & [r_{2n}, r_{2n}] \\
\vdots & \vdots & \vdots \\
[r_{n1}, r_{n1}], [r_{n2}, r_{n2}], & \cdots, & [r_{nn}, r_{nn}]
\end{bmatrix}$$

Construct the optimal scheme:

$$S^+ = \{[r_{11}^+, r_{11}^+], [r_{21}^+, r_{21}^+], \ldots, [r_{nn}^+, r_{nn}^+]\}$$

and get the weight vector of every index:

$$\min d(\omega) = \sum_{i=1}^{n-1} \sum_{j=1}^{n} (r_{ij}^L - r_{ij}^P)^2 + (r_{ij}^U - r_{ij}^Q)^2 \omega_i^2$$

subject to $\sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0, i \in H$

Construct the superiority possibility matrix $P$:

$$P = \begin{bmatrix}
z_1 & P_{11} & P_{12} & \cdots & P_{1n} \\
z_2 & P_{21} & P_{22} & \cdots & P_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
z_n & P_{n1} & P_{n2} & \cdots & P_{nn}
\end{bmatrix}$$

Obtain the superiority possibilities vector $v = (v_1, v_2, \ldots, v_n)$ and choose the optimum scheme.

**VI. EXAMPLES**

**A. Example 1**

A firm decides to produce a new product, through investigation, there are four possibilities, they are separately

- Good sale ($X_1$),
- Regular sale ($X_2$),
- Slightly poor ($X_3$),
- Bad sale ($X_4$),

considering three criteria, they are separately direct gain ($Q_1$), indirect gain ($Q_2$) and lost ($Q_3$), there are three alternatives $A_1, A_2, A_3$ , the attribute value under each criteria is shown by TABLE I [16]. How to find the optimal alternative?

1. Calculating the standard deviation, it is shown by TABLE I.

**TABLE I. THE STANDARD EXPECTED VALUE**

<table>
<thead>
<tr>
<th></th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>3.725,4.772</td>
<td>19.609,23.652</td>
<td>10.868,12.691</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2.846,4.121</td>
<td>28.615,35.399</td>
<td>12.849,15.997</td>
</tr>
<tr>
<td>$S_3$</td>
<td>4.409,6.009</td>
<td>20.412,25.515</td>
<td>8.263,10.188</td>
</tr>
</tbody>
</table>

(2) The normalization value of standard expected value is shown by TABLE II.

**TABLE II. THE NORMALIZED STANDARD EXPECTED VALUE**

<table>
<thead>
<tr>
<th></th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0.620,0.795</td>
<td>0.554,0.668</td>
<td>0.679,0.793</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0.474,0.686</td>
<td>0.808,1</td>
<td>0.803,1</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0.734,1</td>
<td>0.577,0.721</td>
<td>0.517,0.637</td>
</tr>
</tbody>
</table>

(3) Construct the optimal scheme $S^+$, which is shown by TABLE III.

**TABLE III. THE OPTIMAL SCHEME $S^+$**

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>0.4276,0.742</td>
<td>0.5763,0.8795</td>
<td>0.5062,0.9334</td>
</tr>
</tbody>
</table>

(4) Calculating the weighted interval number decision making matrix, suppose the weight has been by expert, $\omega = (0.43,0.37,0.2)$, and the weighted interval number decision making matrix is shown by TABLE IV.

**TABLE IV. THE WEIGHTED STANDARD EXPECTED VALUE**

<table>
<thead>
<tr>
<th></th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0.4276,0.742</td>
<td>0.3951,0.5875</td>
<td>0.476,0.6767</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0.3269,0.6403</td>
<td>0.5763,0.8795</td>
<td>0.563,0.8534</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0.5062,0.9334</td>
<td>0.4115,0.6341</td>
<td>0.3625,0.5436</td>
</tr>
</tbody>
</table>

(5) Construct the superiority possibility matrix $P$:

$$P = \begin{bmatrix}
0.5 & 0.3724 & 0.418 \\
0.6276 & 0.5 & 0.5417 \\
0.582 & 0.4583 & 0.5 \\
\end{bmatrix}$$

(6) The ranking of alternatives is shown by TABLE V.

**TABLE V. THE FINAL RANKING BY OPTIMAL MEMBERSHIP DEGREE**

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal grey correlarive distance [16]</td>
<td>0.938</td>
<td>0.667</td>
<td>0.797</td>
</tr>
<tr>
<td>Non-ideal grey correlarive distance [16]</td>
<td>0.174</td>
<td>0.829</td>
<td>0.668</td>
</tr>
<tr>
<td>Distance-Proximity [16]</td>
<td>0.156</td>
<td>0.554</td>
<td>0.456</td>
</tr>
<tr>
<td>Ideal angle cosine [16]</td>
<td>0.843</td>
<td>0.927</td>
<td>0.92</td>
</tr>
<tr>
<td>Non-ideal angle cosine [16]</td>
<td>0.994</td>
<td>0.952</td>
<td>0.875</td>
</tr>
<tr>
<td>Angle-proximity [16]</td>
<td>0.459</td>
<td>0.494</td>
<td>0.513</td>
</tr>
<tr>
<td>Optimal Membership Degree [16]</td>
<td>0.396</td>
<td>0.528</td>
<td>0.473</td>
</tr>
<tr>
<td>This paper</td>
<td>0.2868</td>
<td>0.3710</td>
<td>0.3423</td>
</tr>
</tbody>
</table>

In the TABLE V, the ranking by Optimal Membership Degree is $A_2 > A_3 > A_1$, the $A_2$ is the optimal alternative;
Meanwhile, we should find, according to distance-Proximity method, the ranking is $A_2 \succ A_3 \succ A_4$, also, $A_2$ is the optimal alternative; but according to angle-proximity method, the ranking is $A_3 \succ A_2 \succ A_4$, the $A_3$ is the optimal alternative, the result obtained by angle-proximity is different with another methods; because the Optimal Membership Degree is base on distance-Proximity method and angle-proximity method, its result is scientific and reasonable, the ranking is $A_2 \succ A_3 \succ A_4$, also, $A_2$ is the optimal alternative. According to the method of this paper, the ranking is $A_2 \succ A_3 \succ A_4$, also, $A_2$ is the optimal alternative, which is more appropriate if the number of alternatives is not too large, which is more explicit and richer to show superiority possibilities between the alternatives.

B. Example 2

To develop new product, five investment alternatives $S_i (i = 1,2,3,4,5)$ are drafted. And decision attributes includes expected net present value, venture profit value, investment amount and risk loss value. Expected net present value and venture profit value are benefit attributes; investment amount and risk loss value are cost attribute. The attribute values of every alternative are showed in Table VI (unit: ten thousand RMB) [17-18].

**TABLE VI. PRODUCT INVESTMENT DECISION DATA**

<table>
<thead>
<tr>
<th></th>
<th>Investment amount ($Q_1$)</th>
<th>Expected net present value ($Q_2$)</th>
<th>Venture profit value ($Q_3$)</th>
<th>Risk loss value ($Q_4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>[5.7]</td>
<td>[4.5]</td>
<td>[4.6]</td>
<td>[0.4,0.6]</td>
</tr>
<tr>
<td>$S_2$</td>
<td>[10,11]</td>
<td>[6.7]</td>
<td>[5.6]</td>
<td>[1,5.2,0.0]</td>
</tr>
<tr>
<td>$S_3$</td>
<td>[5.6]</td>
<td>[4.5]</td>
<td>[3.4]</td>
<td>[0,0.4,0.7]</td>
</tr>
<tr>
<td>$S_4$</td>
<td>[9,11]</td>
<td>[5.6]</td>
<td>[5.7]</td>
<td>[1,3,1.5]</td>
</tr>
<tr>
<td>$S_5$</td>
<td>[6,8]</td>
<td>[3.5]</td>
<td>[3.4]</td>
<td>[0,0.8,1]</td>
</tr>
</tbody>
</table>

(4) Construct the superiority possibility matrix $P$:

$$P = \begin{bmatrix} 0.5 & 1 & 0.6046 & 1 & 0.9386 \\ 0 & 0.5 & 0 & 0.2986 & 0.0336 \\ 0.3954 & 0 & 1 & 0.8972 \\ 0 & 0.7014 & 0 & 0.5 & 0.2484 \\ 0.0614 & 0.9664 & 0.1028 & 0.7516 & 0.5 \end{bmatrix}$$

(5) Calculate the superiority possibilities vector:

$$r = (0.3235, 0.0666, 0.3034, 0.116, 0.1906)$$

(6) Ranking:

According to the value of $r$ the superiority possibilities with respect to every alternative, the rank of alternatives will be:

$$S_1 \succ S_3 \succ S_5 \succ S_4 \succ S_2$$

The rank result obtained by using the method proposed in this paper is accordant with that by using decision methods discussed in literature [17] and [18]. And this shows the validity of the method proposed in this paper enough.

For the attributes of this decision problem are represented by interval number, and there is no more information about attribute values, the attribute value could be regarded as random variable with homogeneous distribution on this interval. And the decision process using the method proposed in this paper is as follows:

1. Normalize interval $[a_{ij}^L,a_{ij}^U]$ into interval $[v_{ij}^L,v_{ij}^U]$, then get:

**TABLE VII. THE NORMALIZED INTERVAL MATRIX R**

<table>
<thead>
<tr>
<th></th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$Q_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>[0.6667,1]</td>
<td>[0.25,0.5]</td>
<td>[0.25,0.75]</td>
<td>[0.875,1]</td>
</tr>
<tr>
<td>$S_2$</td>
<td>[0.0,0.1667]</td>
<td>[0.75,1]</td>
<td>[0.5,0.75]</td>
<td>[0.0,0.3125]</td>
</tr>
<tr>
<td>$S_3$</td>
<td>[0.8333,1]</td>
<td>[0.25,0.5]</td>
<td>[0.0,0.25]</td>
<td>[0.8125,1]</td>
</tr>
<tr>
<td>$S_4$</td>
<td>[0.0,0.3333]</td>
<td>[0.5,0.75]</td>
<td>[0.5,1]</td>
<td>[0.3125,0.4375]</td>
</tr>
<tr>
<td>$S_5$</td>
<td>[0.5,0.8333]</td>
<td>[0.0,0.5]</td>
<td>[0.25]</td>
<td>[0.625,0.75]</td>
</tr>
</tbody>
</table>

2. Construct the optimal scheme $S^+$

$$S^+ = \{[0.875,1],[0.75,1],[0.8333,1],[0.5,1],[0.625,0.8333]\}$$

3. Calculate the weighted interval number decision making matrix in VIII, and suppose the weight has been by expert, $\omega = (0.3122, 0.1218, 0.1656, 0.4014)$.

**TABLE VIII. THE WEIGHTED STANDARD EXPECTED VALUE**

<table>
<thead>
<tr>
<th></th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$Q_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>[0.2075,0.3112]</td>
<td>[0.0305,0.069]</td>
<td>[0.0415,0.1244]</td>
<td>[0.3512,0.4014]</td>
</tr>
<tr>
<td>$S_2$</td>
<td>[0.00519]</td>
<td>[0.0914,0.1218]</td>
<td>[0.0829,0.1244]</td>
<td>[0.01254]</td>
</tr>
<tr>
<td>$S_3$</td>
<td>[0.2593,0.3112]</td>
<td>[0.0305,0.069]</td>
<td>[0.0415]</td>
<td>[0.3261,0.4014]</td>
</tr>
<tr>
<td>$S_4$</td>
<td>[0.0,0.1037]</td>
<td>[0.0609,0.0914]</td>
<td>[0.0829,0.1658]</td>
<td>[0.1254,0.1756]</td>
</tr>
<tr>
<td>$S_5$</td>
<td>[0.1556,0.2593]</td>
<td>[0.0069]</td>
<td>[0.0415]</td>
<td>[0.2509,0.3011]</td>
</tr>
</tbody>
</table>

VII. Conclusion

Multiple attribute decision making under risk problem has wide application in practice. Aiming at multiple attribute decision making problems with continuous random variable on finite interval, this paper proposes a rank method based on superiority possibilities, and presents decision-making steps. On the whole, this method shows clear thought, and is easy to understand, and it can be regarded as the enrichment and development for risk decision theory and methods. In the meantime, this method also provides a new idea for multiple attribute decision making problems.

Multiple attribute decision-making problems under uncertainty take on very important meaning in theory and practice of systems engineering, their benefit and loss matrixes reflect high complexity and uncertainty of the problems, also reflect the risk information of benefit chance and loss chance faced by decision-maker. The method proposed in this paper used the superiority possibilities to solve multiple attribute decision making problems with interval numbers, at first, this paper transformed the multiple attribute decision making problems under uncertainty into a non-risk normalized multiple
attribute decision making matrix, using the established positive ideal reference, a superiority possibilities matrix can be obtained, then the superiority possibilities vector can be used to rank the alternatives, and its result is scientific and reasonable.

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REFERENCES