Ranking Alternatives Expressed With Interval-valued Intuitionistic Fuzzy Set

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Abstract—Interval-valued intuitionistic fuzzy (IVIF) set (IVIFS) is effective in dealing with fuzziness and uncertainty inherent in decision data derived from decision maker and the actions performed in multi-attribute decision making (MADM). Therefore, in this paper, we investigate the MADM problems with uncertain attribute weight information in the framework of IVIFS. We first propose a new accuracy function for solving the problem of ranking alternatives expressed with IVIFSs. Secondly, we propose a linear-programming decision making method based on the accuracy function to solve the MADM problem with binding attribute weight information conditions under IVIF environment. Thirdly, we present an entropy-based decision making method based on the accuracy function to deal with the MADM problem with completely unknown attribute weights under IVIF environment. Finally, two supplier selection examples are given to demonstrate the feasibility and validity of the proposed MADM methods, by comparing it with other fuzzy MADM methods.

I. INTRODUCTION

Atanassov and Gargov extended the intuitionistic fuzzy set (IFS) to the interval-valued intuitionistic fuzzy (IVIF) set (IVIFS) [1], [2], characterized by membership and nonmembership functions whose values are intervals rather than real numbers. As an extension of IFS, IVIFS is much better in dealing with some practical problems with vagueness and uncertainty compared with IFS [3]. Since the IVIFS was proposed, there is lots of literature emphasizing on theoretical research and applied research of IVIFS. Bustince and Burillo [5], Grzegorzewski [6], Chen [7], Xu and Chen [8] systematically investigated various measures (distance, similarity degree, correlation coefficient and so on) on IVIFSs. Szmidt and Kacprzyk [9], Hung [10], Zhang [11], Zeng [12], and Ye [13] constructed various entropies on IFSs/IVIFSs. Yager [14] and Xu [15] proposed some aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOG) operator, the interval-valued intuitionistic fuzzy hybrid geometrical (IIFHG) operator, and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator. Some ranking functions were proposed to solve the problem of ranking alternatives expressed by IVIFSs. Chen and Tan [16] proposed the score function. Xu [17] and Ye [18] proposed two accuracy functions. However, there are some defects in the existing ranking functions for ranking alternatives expressed via IFS/IVIFS. We propose a new accuracy function to overcome some defects of existing ranking functions, which is one key point in this paper. Besides the theoretical works on IVIFSs, lots of literature emphasized on the practical application fields, such as decision making, image edge detection, fuzzy risk analysis and some problems with uncertainty [13], [16], [17], [18], [19], [20], [21]. In particular, IFS and IVIFS have been proved to be effective and feasible for addressing the fuzzy decision making problems [13], [16], [17], [18], [20], [21].

Fuzzy MADM is an important part of decision making and expert system fields. On the one hand, the preferences over alternatives provided by decision makers can not be expressed by crisp values of membership and nonmembership during the action process of decision making for some situations with uncertainty. A decision maker may provide his/her preferences over alternatives with IVIF numbers. All provided preference values can be conveniently contained in a matrix, i.e., an IVIF decision making matrix. On the other hand, the attribute weight information may be unknown or can not be represented by crisp values. In general, the attribute weight information includes the following three cases: (1) crisp attribute weight values, (2) partially known attribute weight information, (3) unknown attribute weight information [20], [21], [22]. So far, lots of MADM literature emphasized on solving the fuzzy decision making problems with crisp attribute weights values [20], [21], [22]. However, how to solve the fuzzy MADM problems with partially known or unknown attribute weight information under IVIF environment remains an open issue in decision making analysis and expert system fields. Therefore, another key point in this paper is to solve the MADM problems with binding attribute weight information conditions and completely unknown attribute weights under IVIF environment.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of IVIFS and some operations. A new accuracy function of IVIFSs is introduced in Section 3. In Section 4, we investigate the MADM problems with partially known and completely unknown attribute weight information under IVIF environment, respectively. Two numerical examples are utilized to illustrate the applicability of the presented approach in Section 5. Finally, a conclusion is given in Section...
6. II. PRELIMINARIES

A. The concept of IVIFS

**Definition 1.** [2] Let \(X\) be a set and \(\text{Int}[0,1]\) be all subintervals of \([0,1]\). An IVIFS \(A\) in \(X\) is defined with the form
\[
A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}
\]
where
\[
\mu_A : X \to \text{Int}[0,1], \nu_A : X \to \text{Int}[0,1]
\]
with the condition
\[
0 \leq \sup(\mu_A(x)) + \sup(\nu_A(x)) \leq 1, \text{ for all } x \in X
\]
The intervals \(\mu_A(x)\) and \(\nu_A(x)\) denote the membership and non-membership degrees of \(x\) to \(A\), respectively. Let
\[
A = \{(x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]) | x \in X\}
\]
where \(\mu_{A_L}(x) = \inf(\mu_A(x)), \nu_{A_L}(x) = \inf(\nu_A(x)), \mu_{A_U}(x) = \sup(\mu_A(x)), \nu_{A_U}(x) = \sup(\nu_A(x))\). If \(\mu_{A_L}(x) = \mu_{A_U}(x)\) and \(\nu_{A_L}(x) = \nu_{A_U}(x)\), then IVIFS \(A\) reduces to an IFS.

B. The operations on IVIFSs

Let IVIFSs \(X\) be all the IVIFSs on \(X\). We introduce the following operations of IVIFSs.

**Definition 2.** [2] Let \(A, B \in \text{IVIFS}(X)\). The operations between \(A\) and \(B\) are as follows:

1. \(A \leq B\) if and only if \(\mu_{A_L}(x) \leq \mu_{B_L}(x), \mu_{A_U}(x) \leq \mu_{B_U}(x), \nu_{A_L}(x) \geq \nu_{B_L}(x), \text{ and } \nu_{A_U}(x) \geq \nu_{B_U}(x)\) for all \(x \in X\).
2. \(A = B\) if and only if \(A \leq B\) and \(B \leq A\) for all \(x \in X\).

**Definition 3.** [15] Let \(\alpha_j = ([a_j, b_j], [c_j, d_j]) (j = 1, 2, \ldots, n)\) be a set of IVIF numbers, then the IIFWA operator is defined as follows:
\[
\text{IIFWA}_\Phi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \frac{1 - (1 - a_j)^{w_j}}{1 - (1 - b_j)^{w_j}},
\]
where \(w_j\) denotes the weight of \(\alpha_j (j = 1, 2, \ldots, n)\), \(w_j \geq 0\) and \(\sum_{j=1}^{n} w_j = 1\).

**Definition 4.** [16] Let \(\alpha = ([a, b], [c, d])\) be an IVIF number. The score function \(s\) of \(\alpha\) is defined as follows:
\[
s(\alpha) = \frac{1}{2} (a + b - c - d)
\]
where \(s(\alpha) \in [0, 1]\).

**Definition 5.** [17] Let \(\alpha = ([a, b], [c, d])\) be an IVIF number. The accuracy function \(h_1\) of \(\alpha\) is defined as follows:
\[
h_1(\alpha) = \frac{1}{2} (a + b + c + d)
\]
where \(h_1(\alpha) \in [0, 1]\).

**Definition 6.** [18] Let \(\alpha = ([a, b], [c, d])\) be an IVIF number. The accuracy function \(h_2\) of \(\alpha\) is defined as follows:
\[
h_2(\alpha) = a + b - \frac{c + d}{2}
\]
where \(h_2(\alpha) \in [-1, 1]\).

III. THE NEW ACCURACY FUNCTION

In this section, we first propose an accuracy function to overcome some defects of existing ranking functions. Furthermore, some illustrative examples are given to show the invalidity of aforementioned ranking functions. The new accuracy function of IVIFS is defined as follows:

**Definition 7.** Let \(\alpha = ([a, b], [c, d])\) be an IVIF number. The accuracy function \(\Phi\) of \(\alpha\) is defined by
\[
\Phi(\alpha) = \frac{a + b - c + d}{a + b + c + d} \cdot \exp\left(-\frac{a + b + c + d}{2}\right)
\]

Note 3.1. The score function \(s\) fails to rank the following two alternatives denoted by IVIF numbers in Example 3.1.

**Example 3.1.** If IVIF numbers for two alternatives are \(\alpha = ([0.4, 0.5], [0.1, 0.2])\) and \(\beta = ([0.3, 0.4], [0.1, 0.2])\), then the desirable alternative is selected according to the score function \(s\). We get \(s(\alpha) = 0.2\) and \(s(\beta) = 0.2\), and do not know which alternative is better for this case.

**Note 3.2.** The accuracy function \(h_1\) fails to rank the following two alternatives denoted by IVIF numbers in Example 3.2.

**Example 3.2.** If IVIF numbers for two alternatives are \(\alpha = ([0.4, 0.5], [0.1, 0.2])\) and \(\beta = ([0.4, 0.5], [0.05, 0.15])\), then the desirable alternative is selected according to the accuracy function \(h_1\). We get \(h_1(\alpha) = 0.6\) and \(h_1(\beta) = 0.6\), and do not know which alternative is better for this case.

**Note 3.3.** The accuracy functions \(h_1\) and \(h_2\) may induce unreasonable results.

**Example 3.3.** If IVIF numbers for two alternatives are \(\alpha = ([0.2, 0.3], [0.4, 0.5])\) and \(\beta = ([0.1, 0.2], [0.7, 0.8])\), then the desirable alternative is selected according to the accuracy function \(h_1\) and \(h_2\). We get \(h_1(\alpha) = 0.7\), \(h_1(\beta) = 0.9\), \(h_2(\alpha) = -0.05\) and \(h_2(\beta) = 0.05\). The results imply that \(\beta\) is better than \(\alpha\) based on \(h_1\) or \(h_2\). According to Definition 2, we get \(a > b\), which contradicts the ranking order derived from \(h_1\) and \(h_2\).

**Note 3.4.** The accuracy function \(h_2\) fails to rank the following two alternatives denoted by IVIF numbers in Example 3.4.

**Example 3.4.** If IVIF numbers for two alternatives are \(\alpha = ([0.4, 0.5], [0.1, 0.3])\) and \(\beta = ([0.35, 0.45], [0.15, 0.45])\), then the desirable alternative is selected according to the accuracy function \(h_2\). We get \(h_2(\alpha) = 0.1\) and \(h_2(\beta) = 0.1\), and do not know which alternative is better for this case.

We utilize the accuracy function \(\Phi\) to calculate the above-mentioned four examples. We get \(\Phi(\alpha) = 0.5476\) and \(\Phi(\beta) = 0.5033\) for Example 3.1, and hence \(\alpha\) is better than \(\beta\). We
get $\Phi(\alpha) = 0.6293$ and $\Phi(\beta) = 0.7293$ for Example 3.2, and hence $\beta$ is better than $\alpha$. We get $\Phi(\alpha) = 0.1476$ and $\Phi(\beta) = -0.2341$ for Example 3.3, and hence $\alpha$ is better than $\beta$. We get $\Phi(\alpha) = 0.5893$ and $\Phi(\beta) = 0.4476$ for Example 3.4, and hence $\alpha$ is better than $\beta$.

The above-mentioned four numerical examples show that the proposed accuracy function $\Phi$ is better than existing score function and accuracy functions for ranking alternatives expressed via IVIFS.

Note 3.5. The accuracy function $\Phi$ fails to rank the comparable IVIF numbers for two alternatives is shown in the following example.

Example 3.5. If IVIF numbers for two alternatives are $\alpha = ([0.4, 0.5], [0.2, 0.3])$ and $\beta = ([0.16, 0.74], [0.24, 0.26])$, we get $\Phi(\alpha) = 0.5476$ and $\Phi(\beta) = 0.5476$. Using the existing ranking functions to rank $\alpha$ and $\beta$, we get $s(\alpha) = 0.2, s(\beta) = 0.2, h_1(\alpha) = 0.7, h_1(\beta) = 0.7, h_2(\alpha) = 0.15, h_2(\beta) = 0.15, \Phi(\alpha) = 0.5476$ and $\Phi(\beta) = 0.5476$. All the ranking results imply that we do not know which alternative is better for this case.

We define a rule of $\Phi$ for ranking alternatives expressed via IVIFSs.

Definition 8. Assume $\alpha$ and $\beta$ be two IVIF numbers.
(a) If $\Phi(\alpha) > \Phi(\beta)$, then $\alpha > \beta$.
(b) If $\Phi(\alpha) = \Phi(\beta)$, then $\alpha \sim \beta$.

where $\alpha \sim \beta$ denotes that $\alpha$ approximately equals to $\beta$.

It is clear that $\Phi$ satisfies the following theorem.

Theorem 1. Let $\alpha$ and $\beta$ be two IVIF numbers. If $\alpha \geq \beta$, then $\Phi(\alpha) \geq \Phi(\beta)$.

Proof. Assume that $\alpha = ([a_i, b_i], [c_i, d_i])$ and $\beta = ([a_2, b_2], [c_2, d_2])$. We introduce a function $f(x, y) = x - y + (x + y)e^{-(x+y)} (0 \leq x, y, x + y \leq 1)$. Since $\frac{\partial f(x, y)}{\partial x} = 1 + (1 - x - y)e^{-(x+y)}$ and $\frac{\partial f(x, y)}{\partial y} = -1 + (1 - x - y)e^{-(x+y)}$, it is clear that $\frac{\partial f(x, y)}{\partial x} > 0$ and $\frac{\partial f(x, y)}{\partial y} < 0$. It means that $f(x,y)$ can be regarded as an increasing function of $x$, and a decreasing function of $y$. Since $\alpha \geq \beta$, it implies that

$$\frac{a_1 + b_1}{2} \geq \frac{a_2 + b_2}{2} \text{ and } \frac{c_1 + d_1}{2} \leq \frac{c_2 + d_2}{2}.$$ Basing on the properties of function $f(x, y)$, we have

$$\frac{a_1 + b_1}{2} - \frac{c_1 + d_1}{2} \geq \frac{a_2 + b_2}{2} - \frac{c_2 + d_2}{2}.$$ (10)

i.e., $\Phi(\alpha) \geq \Phi(\beta)$.

The proof is completed.

IV. MADM UNDER IVIF ENVIRONMENT

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of alternatives and $X = \{X_1, X_2, \ldots, X_k\}$ be a set of attributes. The IVIF decision matrix $D$ of $A$ on $X$ is as follows:

$$D = \begin{bmatrix}
    \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\
    \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nk}
\end{bmatrix}$$ (11)

where $\alpha_{ij}$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, k$) is an IVIF number.

Generally speaking, the attribute weight information is uncertain or fuzzy for actual decision making problems. So far, solving the fuzzy MADM problems with uncertain attribute weights in the framework of IVIFS remains an open problem. In the following part, we emphasize on solving the decision making problem with binding attribute weight information conditions and the decision making problem with completely unknown attribute weights, respectively.

A. IVIF MADM problem with binding attribute weight information conditions

Assume that the attribute weight information is denoted by the binding conditions $\Lambda$ for the aforementioned MADM problem under IVIF environment. And then, we present a linear programming method based on the newly accuracy function to determine the attribute weights.

First of all, we define the accuracy matrix as follows:

Definition 9. Let $D_{n \times k}$ be an IVIF decision making matrix. Then we call $H = [\Phi_{ij}]_{n \times k}$ an accuracy matrix of $D_{n \times k}$, where $\Phi_{ij}$ is the accuracy value of $\alpha_{ij}$.

According to the accuracy matrix, we present the overall accuracy values of each alternatives $A_i (i = 1, 2, \ldots, n)$:

$$\Phi_i(w) = \sum_{j=1}^{k} w_j \Phi_{ij}$$ (12)

where $w = [w_1, w_2, \ldots, w_k]^T$ denotes the attribute weight vector and $\sum_{j=1}^{k} w_j = 1 (w_j \geq 0)$. Obviously, the greater the value $\Phi_i(w)$, the better the alternative $A_i$. For certain alternative $A_i (i = 1, 2, \ldots, n)$, the reasonable attribute weight vector $w$ should be determined by the following optimization model.

$$\begin{array}{ll}
\text{Max} & \Phi_i(w) \\
\text{Subject to :} & w \in \Lambda, w_j \geq 0, \sum_{j=1}^{k} w_j = 1
\end{array}$$ (13)

Based on Eq. (13), we get the optimal solution

$$w^{(i)} = [w^{(i)}_1, w^{(i)}_2, \ldots, w^{(i)}_k]^T (i = 1, 2, \ldots, n)$$ (14)

to the alternative $A_i$. However, in the process of determining the weight vector $w = [w_1, w_2, \ldots, w_k]^T$, we need to consider all the alternatives $A_i (i = 1, 2, \ldots, n)$ as a whole. Let $W = [w^{(1)}, w^{(2)}, \ldots, w^{(n)}]_{k \times n}$, i.e.,

$$W = \begin{bmatrix}
    w^{(1)}_1 & w^{(2)}_1 & \cdots & w^{(n)}_1 \\
    w^{(1)}_2 & w^{(2)}_2 & \cdots & w^{(n)}_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    w^{(1)}_k & w^{(2)}_k & \cdots & w^{(n)}_k
\end{bmatrix}$$ (15)

Let $\Upsilon = (HW)^T (HW)$, and $\varpi = [\varpi_1, \varpi_2, \ldots, \varpi_n]^T$ be the normalized eigenvector of $\Upsilon$. We define the attribute weight vector as

$$w = W \varpi = w^{(1)}_1 + \varpi_2 w^{(2)} + \cdots + \varpi_n w^{(n)}$$ (16)

satisfying $\sum_{j=1}^{k} w_j = 1$ and $w_j \geq 0$. 2319
B. IVIF MADM problem with completely unknown attribute weight information

In this part, we emphasize on assessing the attribute weights for the MADM problem with unknown attribute weights in the framework of IVIFS. The process of determining the attribute weights is as follows:

Step 1. Calculate the accuracy matrix $H$ of $D$.

Step 2. Normalize the accuracy matrix $H$ as follows:

$$H = \left[ \begin{array}{ccc} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1k} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \cdots & \Phi_{nk} \end{array} \right]$$

(17)

where $\Phi_{ij} = \frac{\mid \Phi_{ij} \mid}{\sum_{j=1}^{n} \mid \Phi_{ij} \mid}$ ($i = 1, \ldots, n; j = 1, \ldots, k$).

Step 3. Determine the attribute weights.

Let

$$E_j = -\frac{1}{\ln n} \sum_{i=1}^{n} \Phi_{ij} \ln \Phi_{ij}, \quad j = 1, \ldots, k$$

(18)

Furthermore, the attribute weight $w_j$ ($j = 1, \ldots, k$) is defined by

$$w_j = \frac{1 - E_j}{\sum_{j=1}^{k} (1 - E_j)}$$

(19)

C. MADM method under IVIF environment

Basing on the newly accuracy function $\Phi$ and the IIFWA operator, the MADM method under IVIF environment is as follows.

Step 1. Determine the attribute weight vector $w = [w_1, w_2, \ldots, w_k]$.

Step 2. Calculate

$$\alpha_i = \text{IIFWA}_w(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik}), \quad i = 1, 2, \ldots, n$$

(20)

Step 3. Calculate $\Phi(\alpha_i)$ ($i = 1, 2, \ldots, n$).

Step 4. Rank all the alternatives depending on $\Phi(\alpha_i)$ ($i = 1, 2, \ldots, n$), and select the most desirable one.

V. NUMERICAL EXAMPLES

Example 5.1. There is a supplier selection problem concerning a manufacturing company with the following four possible potential alternatives: $A_1$, $A_2$, $A_3$ and $A_4$. The supplier selection must take a decision according to the following five attributes: $X_1$ (Cost of the product), $X_2$ (Quality of the product), $X_3$ (Service performance of supplier), $X_4$ (Supplier’s profile) and $X_5$ (Risk factor). The decision making matrix $D$ of $A_i$ ($i = 1, 2, 3, 4$) on $X_j$ ($j = 1, 2, 3, 4, 5$) is listed in Table I and Table II. Assume that the constraint condition set of attribute weights is denoted by $\Lambda = \{w_1 \leq 0.3, 0.2 \leq w_2 \leq 0.5, 0.1 \leq w_3 \leq 0.2, w_4 \leq 0.4, w_5 \geq w_5 - w_4, w_4 \geq w_1, w_3 - w_1 \leq 0.1, 0.1 \leq w_4 \leq 0.3\}$.

In the following part, we present the decision making process as follows:

Step 1. Determine the attribute weights.

Firstly, calculate the accuracy matrix $H$ of $D$ as follows:

$$H = \begin{bmatrix} 0.6007 & 0.3331 & 0.8612 & 0.7443 & 0.1333 \\ 0.3849 & 0.1623 & 0.3119 & 0.0173 & 0.8986 \\ 0.4220 & 0.9045 & 0.8549 & 0.6158 & 0.7262 \\ 0.3154 & -0.1674 & -0.1281 & 0.1287 & 0.0495 \end{bmatrix}$$

(21)

According to Eq. (13), we get

$$W = \begin{bmatrix} 0.2667 & 0.1600 & 0.1000 & 0.3000 \\ 0.1000 & 0.1000 & 0.2000 & 0.1000 \\ 0.3667 & 0.2600 & 0.2000 & 0.2000 \\ 0.2667 & 0.1600 & 0.2500 & 0.3000 \\ 0.0000 & 0.3200 & 0.2500 & 0.1000 \end{bmatrix}$$

(22)

Secondly, calculate the normalized eigenvector $\omega$ of $T$ as follows.

$$\omega = [0.0002 \ 0.0011 \ 0.0140 \ 0.9847]^T$$

(23)

Furthermore, we get the attribute weight vector

$$w = [0.2970 \ 0.1014 \ 0.2001 \ 0.2991 \ 0.1023]^T$$

(24)

Step 2. Based on the IIFWA operator and the attribute weight vector $w$, we get a set of IVIF ability values $\alpha_i$ ($i = 1, 2, 3, 4$) for every alternative $A_i$ ($i = 1, 2, 3, 4$).

$$\alpha_1 = [(0.4612, 0.6128), (0.1488, 0.2920)]$$

(25)

$$\alpha_2 = [(0.3208, 0.4508), (0.3350, 0.4819)]$$

(26)

$$\alpha_3 = [(0.4930, 0.6525), (0.1869, 0.3092)]$$

(27)

$$\alpha_4 = [(0.2043, 0.3227), (0.4260, 0.5961)]$$

(28)

Step 3. The accuracy values $\Phi(\alpha_i)$ ($i = 1, 2, 3, 4$) are as follows:

$$\Phi(\alpha_1) = 0.6717$$

(29)

$$\Phi(\alpha_2) = 0.3362$$

(30)

$$\Phi(\alpha_3) = 0.6731$$

(31)

$$\Phi(\alpha_4) = 0.1094$$

(32)
**TABLE III**

<table>
<thead>
<tr>
<th>MADM method</th>
<th>Ranking order of $A_i$ ($i = 1, 2, 3, 4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Park’s method[22]</td>
<td>$A_3 &gt; A_4 &gt; A_2 &gt; A_1$</td>
</tr>
<tr>
<td>Xu’s method[24]</td>
<td>$A_4 &gt; A_1 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>Zhang’s method[25]</td>
<td>$A_3 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
<tr>
<td>Our method</td>
<td>$A_5 &gt; A_3 &gt; A_1 &gt; A_2 &gt; A_4$</td>
</tr>
</tbody>
</table>

**TABLE IV**

<table>
<thead>
<tr>
<th>IVIF decision matrix $D$ of Example 5.2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.2, 0.5), (0.4, 0.5)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.3, 0.4), (0.3, 0.5)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.4, 0.5), (0.2, 0.3)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.4, 0.7), (0.0, 0.2)</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(0.6, 0.7), (0.1, 0.3)</td>
</tr>
</tbody>
</table>

Firstly, calculate the accuracy matrix $H$ of $D$.

$$H = \begin{bmatrix}
0.2595 & 0.3476 & 0.2133 & 0.3133 & 0.1133 & 0.2595 \\
0.3043 & 0.8595 & 0.6133 & 0.6595 & 0.5133 & 0.8133 \\
0.5476 & 1.0595 & 0.8133 & 0.9659 & 0.2595 & 0.7476 \\
0.7893 & 1.3174 & 1.0595 & 1.1659 & 0.7133 & 0.9133 \\
0.8133 & 0.6893 & 1.0595 & 0.4293 & 1.1659 & 0.4595 \\
\end{bmatrix}$$

Furthermore, calculate the normalized matrix $\bar{H}$ based on $H$.

$$\bar{H} = \begin{bmatrix}
0.0956 & 0.0813 & 0.0567 & 0.0887 & 0.0410 & 0.0813 \\
0.1121 & 0.2011 & 0.1632 & 0.1866 & 0.1856 & 0.2547 \\
0.2018 & 0.2479 & 0.2164 & 0.2733 & 0.0938 & 0.2341 \\
0.2908 & 0.3083 & 0.2819 & 0.3299 & 0.2579 & 0.2860 \\
0.2997 & 0.1613 & 0.2819 & 0.1215 & 0.4216 & 0.1439 \\
\end{bmatrix}$$

Secondly, we get the attribute weight vector as follows:

$$w = [0.1705, 0.1736, 0.1687, 0.1688, 0.1449, 0.1736]^T$$

Step 2. Based on the IIFWA operator and the attribute weight vector $w$, we get a set of IVIF ability values $\alpha_i$ ($i = 1, 2, 3, 4, 5$) for every alternative $A_i$ ($i = 1, 2, 3, 4, 5$).

$$\alpha_1 = (0.2699, 0.4380, [0.3930, 0.5093])$$

$$\alpha_2 = (0.4770, 0.6385, [0.1787, 0.3338])$$

$$\alpha_3 = (0.5622, 0.6689, [0.0000, 0.2323])$$

$$\alpha_4 = (0.7051, 1.0000, [0.0000, 0.0000])$$

$$\alpha_5 = (0.5639, 0.6915, [0.0000, 0.2083])$$

Step 3. The accuracy values $\Phi(\alpha_i)$ ($i = 1, 2, 3, 4, 5$) are as follows:

$$\Phi(\alpha_1) = 0.2616$$

$$\Phi(\alpha_2) = 0.6622$$

$$\Phi(\alpha_3) = 0.8514$$

$$\Phi(\alpha_4) = 1.2160$$

$$\Phi(\alpha_5) = 0.8756$$

Step 4. Since $\Phi(\alpha_5) > \Phi(\alpha_4) > \Phi(\alpha_3) > \Phi(\alpha_2) > \Phi(\alpha_1)$, the ranking order of all the alternatives is $A_4 > A_5 > A_3 > A_2 > A_1$, and the most desirable alternative is $A_4$.

By applying the methods adapted from [13], [23], [24], [25] to Example 5.2, the results are shown in TABLE VII.

TABLE VII shows that the ranking order of all the alternatives of Example 5.2 is the same as the results of [13], [23].
TABLE VII
RANKING ORDER OF ALL THE ALTERNATIVES

<table>
<thead>
<tr>
<th>MADM method</th>
<th>Ranking order of $A_i$ ($i = 1, 2, 3, 4, 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xu’s projection method[23]</td>
<td>$A_1 ≻ A_2 ≻ A_3 ≻ A_4 ≻ A_5$</td>
</tr>
<tr>
<td>Ye’s method[13]</td>
<td>$A_1 ≻ A_2 ≻ A_3 ≻ A_4 ≻ A_5$</td>
</tr>
<tr>
<td>Zhang’s method[24]</td>
<td>$A_1 ≻ A_2 ≻ A_3 ≻ A_4 ≻ A_5$</td>
</tr>
<tr>
<td>Zhang’s method[25]</td>
<td>$A_1 ≻ A_2 ≻ A_3 ≻ A_4 ≻ A_5$</td>
</tr>
<tr>
<td>Our method</td>
<td>$A_1 ≻ A_2 ≻ A_3 ≻ A_4 ≻ A_5$</td>
</tr>
</tbody>
</table>

[24], [25]. The results show that the proposed is effective and feasible in dealing with the IVIF MADM problems with completely unknown attribute weights. Especially, this MADM method provides an effective approach to assess the attribute weights based on the divergence among attributes using information entropy theory.

VI. CONCLUSION

In this paper, we have focused on solving the IVIF MADM problem with uncertain attribute weights. We first introduced a new accuracy function for solving the problem of ranking alternatives expressed with IVIFS. Subsequently, we present a decision making method for solving the MADM problem with binding attribute weight conditions, and the MADM problem with completely unknown attribute weights in the framework of IVIFS. The proposed method can be utilized to solve fuzzy and uncertain decision-making problems derived from supplier selection, public risk, medical diagnosis, and other problems in any aspects.

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