Tensor Decomposition Based $R$-Dimensional Matrix Pencil Method

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Abstract—In this paper, we extend the standard matrix pencil (MP) method to R-dimensional ($R$-D) tensor based MP. Higher-order singular value decomposition (HOSVD) is used to obtain the signal subspace. Performance of tensor based MP method is evaluated by computer simulations. Comparing with the conventional matrix based MP methods, better performance is obtained for tensor based $R$-D MP methods by exploiting the structure of the measurement data. Furthermore, it is straightforward to extend the proposed $R$-D tensor MP to other MP type methods, such as $R$-D unitary tensor MP, $R$-D beamspace tensor MP.

Index Terms—Matrix pencil (MP), $R$-dimensional ($R$-D) tensor MP, tensor decomposition, higher-order singular value decomposition (HOSVD)

I. INTRODUCTION

In the last ten years, interested in tensor decompositions has expanded to many fields, such as data mining, computer vision, signal processing and elsewhere. A state of the art comprehensive review on tensor decompositions is provided in [1], which covers an overview of higher order tensor decompositions, their applications and available software. A tensor is a multiple-dimensional array. Vector and matrix are first-order and second-order tensors, respectively. Tensors of order three or higher are called higher order tensors. Tensor based signal processing offers fundamental advantages over the conventional matrix based methods [2]. The structure of the measurement data is exploited by using tensor based techniques. Furthermore, improved subspace estimation is obtained for tensor decompositions based parameter estimation schemes.

Estimation of signal parameter via a rotational invariant technique (ESPRIT) [3] and matrix pencil (MP) [4] are two widely used for harmonic retrieval (HR) and direction of arrival (DOA) estimation methods, which are subspace based high resolution searching-free approaches by exploiting the shift invariant structure of the signal subspace.

Recently, the conventional matrix based ESPRIT is extended to tensor based ESPRIT. $R$-D standard tensor ESPRIT and $R$-D unitary tensor ESPRIT are proposed in [2]. A preprocessing scheme called tensor based spatial smoothing (TB-SS) is proposed for DOA estimation in [5]. Better performance is obtained for TB-SS when applied in conjunction with tensor ESPRIT type methods. Analytical performance evaluation of HOSVD based subspace estimation schemes is studied in [6]. Furthermore, base on the analytical analysis in [6], performance analysis of tensor ESPRIT type algorithms is discussed in [7].

Comparing with the ESPRIT type methods, MP is applied on the data directly without forming a covariance matrix, which can be processed in real time non-stationary environment. Since it is done on a sample by sample basis, fewer samples are needed. The standard Matrix pencil method is extended to two-dimensional (2-D) MP for two-dimensional frequency estimation in [8]. Three-dimensional (3-D) MP is proposed in [9] to simultaneously estimate azimuth, elevation angles and frequency of the incoming signals. To significantly reduce the computational complexity of MP, unitary matrix pencil (UMP) is proposed for DOA estimation in [10]. The UMP method is extended to two-dimensional UMP.
case to estimate the azimuth and elevation angles of the signals in [11]. Recently, beamspace matrix pencil (BMP) and multiple invariance BMP (MI-BMP) are discussed in [12]. The computationally efficient two-dimensional BMP method is proposed for DOA estimation using a uniform rectangular array in [13].

For the matrix based standard MP methods, the structure information of the measurement data is not fully exploited. Furthermore, it is difficult to process a higher order measurements using matrix based MP. Motivated by the tensor based ESPRIT algorithms, in this paper, we extend the standard MP method to R-D tensor based MP method. Higher-order singular value decomposition is used to obtain the tensor based signal subspace. The proposed R-D tensor MP method can be extended to other MP type methods, such as R-D unitary tensor MP, R-D Beamspace tensor MP. Furthermore, simulation results of tensor based MP method are also given. Comparing with the conventional matrix based MP methods, better performance is obtained for tensor based R-D matrix pencil (MP) methods by exploiting the structure of the measurement data.

The rest of the paper is organized as follows. Signal model is described in Section II. Tensor based MP is explained in Section III. In Section IV, we evaluate the performance of the tensor based MP method. Simulation results are given. Conclusions are drawn in Section V.

Notation

Scalars are denoted by italic letters \((a, b)\), column vectors by lower-case boldface letters \((\mathbf{a}, \mathbf{b})\), matrices by upper-case boldface letters \((\mathbf{A}, \mathbf{B})\) and tensors by calligraphic bold-face letters \((\mathcal{A}, \mathcal{B})\). The \((i, j, k)\)th element of a third order tensor \(\mathcal{A}\) is denoted as \(a_{i,j,k}\). Blackboard bold letters \(\mathbb{R}\) and \(\mathbb{C}\) denote the real and complex numbers, respectively. \((\cdot)^T\), \((\cdot)^*\) and \((\cdot)^H\) denote transpose, conjugation and Hermitian transpose, respectively. \(\mathbf{I}_p\) and \(\mathbf{0}_{p \times q}\) denote a \(p \times p\) identity matrix and a \(p \times q\) zero matrix, respectively.

The tensor operations used in this paper are consistent with [2]. The \(N\)-directional tensor is denoted by \(\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}\), where \(I_n\) is the dimension of the \(n\)th mode of the tensor. The \(n\)th mode vectors of \(\mathcal{A}\) are obtained by varying the \(r\)th index and keeping all other indices fixed.

\(a)\) The \(n\)-mode product: The \(n\)-mode product of a tensor \(\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}\) and a matrix \(\mathbf{U} \in \mathbb{C}^{I_n \times I}\) along the \(n\)th mode is denoted as

\[
\mathcal{B} = \mathcal{A} \times_n \mathbf{U} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_{n-1} \times I_n \times I_{n+1} \times \cdots \times I_N}.
\] (1)

The element of tensor \(\mathcal{B}\) is computed by

\[
b_{i_1,i_2,\ldots,i_{n-1},i_{n+1},\ldots,i_N} = \sum_{i_n=1}^{I_n} b_{i_1,i_2,\ldots,i_{n-1},i_{n+1},\ldots,i_N} \cdot u_{j_n,i_n}.
\] (2)

\(b)\) The higher-order SVD (HOSVD): The higher-order singular value decomposition of a tensor \(\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}\) is given by

\[
\mathcal{A} = \mathcal{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \cdots \times_N \mathbf{U}_N,
\] (3)

where \(\mathcal{S} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}\) is the core tensor which satisfies the all orthogonality conditions and \(\mathbf{U}_n \in \mathbb{C}^{I_n \times I}\), \(n = 1, 2, \cdots, N\), are the unitary matrices of \(n\) mode singular vectors [14] [15].

II. SIGNAL MODEL

The measurement data is modeled as a superposition of \(d\) undamped exponentials from narrowband sources, sampled on an \(R\)-dimensional grid of \(M = M_1 \times M_2 \times \cdots \times M_R\) sensors and observed at \(N\) subsequent time instants. The number of sensors along the \(r\)th dimension is denoted by \(M_r\). The superposition of \(d\) undamped exponentials is described by

\[
x_{m_1,m_2,\ldots,m_R,n} = \sum_{k=1}^{d} s_{k,n} \left( \prod_{r=1}^{R} e^{j(m_r-1)\mu_k^{(r)}} \right) + w_{m_1,m_2,\ldots,m_R,n}
\] (4)

where \(m_r = 1, 2, \cdots, M_r\) and time instant \(n = 1, 2, \cdots, N\). \(s_{k,n}\) is the complex amplitude of the \(k\)th source at time instant \(n\) and \(\mu_k^{(r)}\) is the spatial frequency of the \(k\)th source along the \(r\)th dimension. Noise samples \(w_{m_1,m_2,\ldots,m_R,n}\) are assumed to be i.i.d. zero mean circularly symmetric complex Gaussian random variables and mutually uncorrelated in all the dimensions.

The measurement tensor \(\mathcal{X} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R \times N}\) can be modeled through

\[
\mathcal{X} = \mathcal{A} \times_{R+1} \mathbf{S}^T + \mathcal{W},
\] (5)

where \(\mathcal{A} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R \times d}\) is the array steering tensor, \(\mathbf{S} \in \mathbb{C}^{d \times N}\) is the complex amplitude of the
Let us compare the difference between tensor based and matrix based for 2-D case by letting $R = 2$, then (8) becomes a tensor based 2-D MP method. A 4th order tensor $\mathbf{Y}_2 \in \mathbb{C}^{M_{sub,1} \times M_{sub,2} \times L_1 \times L_2}$ is used to do the estimation. However, for 2-D matrix based MP method using single sample, after 2-D spatial smoothing, the measurement is packed into a augmented matrix $\mathbf{Y}_2 \in \mathbb{C}^{M_{sub,1} L_1 \times M_{sub,2} L_2}$.

2) Multiple Measurements: For multiple measurements, time instant $n = 1, 2, \cdots, N$, $R + 1$ dimensional measurement tensor $\mathbf{X} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R \times N}$. To apply the $R$-D tensor MP method, $\mathbf{X}$ is reformulated as a $2R + 1$ dimensional tensor $\mathbf{Y}$ by using tensor based spatial smoothing. The reformulated tensor is denoted as

$$
\mathbf{Y} \in \mathbb{C}^{M_{sub,1} \times M_{sub,2} \times \cdots \times M_{sub,r} \times L_1 \times L_2 \times \cdots \times L_R \times N}. 
$$

Consider the 1-D case ($R = 1$), for matrix based MP method, the $N$ samples are packed in a $\mathbf{Y}_1 \in \mathbb{C}^{M_{sub,1} \times N L_1}$ augmented matrix. Meanwhile, a 3rd order tensor $\mathbf{Y} \in \mathbb{C}^{M_{sub,1} \times L_1 \times N}$ is used to stack the data for tensor based MP.

B. Tensor based signal subspace

The tensor $\mathbf{Y}$ can be expressed in terms of HOSVD in the following way.

$$
\mathbf{Y} \approx \mathbf{C} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_R \mathbf{U}_R
$$

$$
\times_{R+1} \mathbf{V}_1 \times_{R+2} \mathbf{V}_2 \times_{R+3} \cdots \times_{2R} \mathbf{V}_R \times_{2R+1} \mathbf{D},
$$

where $\mathbf{C}$ is the core tensor, $\mathbf{U}_r \in \mathbb{C}^{M_{sub,r} \times d}$, $\mathbf{V}_r \in \mathbb{C}^{L_r \times d}$ and $\mathbf{D} \in \mathbb{C}^{N \times N}$. Since both $\mathbf{U}_r$ and $\mathbf{V}_r$ contain the subspace information of the $r$th dimension of the signal. To reduce the computational complexity, only one of them is used to obtain the signal subspace. If $M_{sub,r} \geq L_r$, then $\mathbf{U}_r$ is used, otherwise we use $\mathbf{V}_r$.

Let $\mathbf{A}_r \in \mathbb{C}^{M_{sub,r} \times d}$ denote the $r$th mode of the array manifold. Since $\mathbf{A}_r$ and $\mathbf{U}_r$ span the same subspace.

$$
\mathbf{U}_r = \mathbf{A}_r \mathbf{T}_r, \quad r = 1, 2, \cdots, R,
$$

where $\mathbf{T}_r \in \mathbb{C}^{d \times d}$ is a nonsingular complex matrix.

To remove the last row or the first row of $\mathbf{A}_r$ and $\mathbf{U}_r$. Two selection matrices along the $r$th dimension are
defined as
\[
J_1^{(r)} = \begin{bmatrix} I_{M_{sub,r}-1} & 0_{(M_{sub,r}-1)\times 1} \end{bmatrix},
\]
\[
J_2^{(r)} = \begin{bmatrix} 0_{(M_{sub,r}-1)\times 1} & I_{M_{sub,r}-1} \end{bmatrix},
\]
where \( J_1^{(r)} \) and \( J_2^{(r)} \) are elements of \( \mathbb{R}^{(M_{sub,r}-1)\times M_{sub,r}} \), \( r = 1, 2, \ldots, R \).

In order to apply R-D tensor MP, the R dimensional array must satisfy shift invariance in each mode. The R shift invariance equations are expressed as:
\[
J_1^{(r)} A_r \Psi_r = J_2^{(r)} A_r,
\]
where \( \Psi_r \) is a diagonal matrix and
\[
\Psi_r = \begin{bmatrix} e^{j\mu_{1}^{(r)}} & 0 & & \cdots & 0 \\
0 & e^{j\mu_{2}^{(r)}} & & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
0 & & \cdots & e^{j\mu_{d}^{(r)}} & 0 \end{bmatrix}.
\]

For noisy case, we obtain the following approximated R shift invariance equations
\[
J_1^{(r)} U_r \Phi_r \approx J_2^{(r)} U_r.
\]
\( \Psi_r \) and \( \Phi_r \) are related through the eigenvalue reserving transformation
\[
\Phi_r = T_r \Psi_r T_r^{-1}.
\]
The estimation of \( e^{j\mu_{k}^{(r)}} \), \( k = 1, 2, \cdots, d \) are obtained from the eigenvalues of \( \Phi_r \). LS or TLS can be used to solve the linear equation (16).

The final step of R-D tensor MP method is joint eigenvalue estimation. Similar to the R-D tensor ESPRIT type algorithms [2], after computing the eigenvalues of \( \Phi_r \), joint Schur decomposition [17] or simultaneous diagonalization [18] is needed to guarantee the correct paring of the sources over all the R modes. The tensor based MP method is summarized in Algorithm 1.

IV. SIMULATION RESULTS

To evaluate the tensor based MP method compared to the standard matrix based MP method. The following numerical simulations are presented. A uniform linear array (ULA) formed by 5 omni-directional sensors with half-wavelength inter-element spacing is used to estimate two source directions. \( M_1 = 5 \) and \( M_{sub,1} = 3 \). Additive white Gaussian noise (AWGN) is added to the simulated sensor outputs. The proposed algorithm operates on 10 sensor outputs to obtain DOA estimates. The performance of the two methods is evaluated using root mean square error (RMSE). Matlab tensor toolbox [19] is used for all tensor based computations. For matrix based MP method, the measurements are reformulated in \( Y \in \mathbb{C}^{3 \times (3 \times 10)} \). Meanwhile, the measurements are packed in a 3-D tensor \( \mathbf{Y} \in \mathbb{C}^{3 \times 3 \times 10} \).

In the first simulation, the array acquires data from two closely spaced narrowband sources situated at 60° and 65° direction with respect to the array broadside. Fig.1 shows the spatial spectrum of the proposed tensor based MP technique averaged over 500 independent trials in comparison with the MP method. To evaluate the performance of tensor MP for well separated sources, the second simulation is performed. The simulation scenario is same as the previous, except that the two narrowband sources are from 60° and 70°. Fig.2 shows the spatial spectrum of the proposed tensor based MP technique averaged over 500 trials in comparison with the MP method. From Fig.1 and Fig.2, we can see that, compared to the matrix based MP method, better performance is obtained for tensor based MP method with regard to RMSE for both the closely spaced and well separated cases.

V. CONCLUSION

In this paper, we extend the standard MP method to R-D tensor based MP. HOSVD is used to obtain the signal

\[
\text{Algorithm 1: R-D Tensor Matrix Pencil Method}
\]

Obtain the tensor based measurement \( \mathbf{X} \)

for \( r = 1 \) to \( R \) do

Calculate the \( r \)th mode signal subspace \( U_r \)

Define two selection matrices \( J_1^{(r)} \) (12) and \( J_2^{(r)} \) (13).

Obtain \( \Phi_r \) by solving (16) using LS or TLS.

Compute the \( d \) eigenvalues of \( \Phi_r \).

Estimate \( \mu_{k}^{(r)} \), \( k = 1, 2, \cdots, d \) in (4) from the \( k \)th eigenvalues of \( \Phi_r \).

end

Paring the sources over all the \( R \) modes.
subspace. Furthermore, performance of tensor based MP method is evaluated by computer simulations. Comparing with the conventional matrix based MP methods, better performance is obtained for tensor based R-D MP methods by exploiting the structure of the measurement data.

REFERENCES

