Distributed Fusion Filter for Asynchronous Multi-rate Multi-sensor Non-uniform Sampling Systems

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Abstract—This paper is concerned with the distributed fusion filtering problem for a class of asynchronous multi-rate multi-sensor non-uniform sampling discrete stochastic systems, where the state is updated at the highest sampling rate and different sensors may have different lower measurement sampling rates. Furthermore, the state is updated uniformly and the measurement is sampled non-uniformly. The proposed algorithm can significantly improve the estimation accuracy compared to the previous modeling method which ignores the system noise. The simulation research verifies the effectiveness of the proposed algorithm.

Keywords—multi-rate, multi-sensor, non-uniform sampling, covariance intersection, distributed fusion filter

I. INTRODUCTION

Recently, the state estimation problem for multi-sensor multi-rate systems has attracted lots of attention due to the impossibility to sample all the physical signals at one single rate in many complicated practical systems. Generally speaking, there are two cases for it. The first case is the estimation problem for single sensor system [1-2], which includes two rates or four rates as follows: the state updating rate, the measurement sampling rate, the estimation updating rate and the state estimation output rate. The difficulty is how to obtain the state estimation values at no measurement times. The linear minimum variance optimal state filter is presented by state augmentation approach [2]. However, the proposed filter has expensive computational cost due to the high dimension augmented state. The other case is the estimation problem for multi-sensor system [3-16], where the state is updated at the fastest rate and different sensors have different measurement sampling rates. Further, there is at least one sensor with the sampling rate same with updating rate. The difficulty is how to transform the fusion estimation problem for multi-rate systems to the equivalent fusion estimation problem for a single rate system. Generally, there are two methods: multiscale theory [3-7] and Kalman filter [8-16]. Based on multiscale theory, using wavelet transformation, Hong [3-4], Zhang [5] and Wen [6-7] et al. propose some fusion strategies for different sensors with the ratio of the sampling rates being a power of two. However, the white noise is converted into colored noise in the state decomposition [3-4], which results in the suboptimal state estimators. The state augmentation method is used in [5-7], which results in the high dimension computational cost. Based on Kalman filter, Ge [8], Yan and Zhou [9-13] propose some fusion algorithms for different sensors with the ratio of the sampling rates being any positive rational. However, the filters in [8-9] have expensive computational cost, the filter in [10] is suboptimal since some available sensor information is ignored at some instants and the filters in [11-13] are also suboptimal since the system noise is ignored. Moreover, the centralized and distributed asynchronous fusion algorithms are presented for continuous stochastic system [14]. The $H_\infty$ filtering problem is also investigated for multi-rate system [15-16].

Motivated by the above discussion, we consider the distributed fusion filtering problem for an asynchronous multi-rate multi-sensor non-uniform sampling discrete stochastic system. Firstly, the non-augmented state models at each sensor are established by considering the system noises. Then, based on the established state space models, the local filters at the measurement sampling points and at the state update points are proposed. The corresponding filtering error covariance matrices are derived. Using the covariance intersection fusion algorithm, the distributed fusion filter is given based on the local filters and the local filtering error covariance matrices. The proposed algorithm can significantly improve the estimation accuracy compared to the previous modeling method which ignores the system noise. The simulation research verifies the effectiveness of the proposed algorithm.

Consider the following asynchronous multi-rate multi-sensor non-uniform sampling discrete time invariant linear stochastic system with $L$ sensors

$$x(t+1) = \Phi x(t) + \Gamma w(t)$$  \hspace{1cm} (1)

$$y_i(k) = H_i x(k) + v_i(k), \quad i = 1, 2, \cdots, L$$  \hspace{1cm} (2)
where $x(t) \in \mathbb{R}^n$ is the system state at $tT$ time, $T$ is the sampling period, $y_i(k) \in \mathbb{R}^{m_i}$, $i = 1, 2, \ldots, L$ are the $k$th measurements. $\chi_i(k), i = 1, 2, \ldots, L$ are the states measured by $y_i(k)$. $w(t) \in \mathbb{R}$ and $v_i(k) \in \mathbb{R}^{m_i}$ are the system noise and measurement noise. $\Phi_i, \Gamma_i$ and $H_i$ are constant matrices with suitable dimensions. The system state $x(t)$ is updated at the highest rate $S_1$, the measurements are sampled at the lower rates $S_i$, and satisfies $S_j : S_j = n_j$ where $n_j, i = 1, 2, \ldots, L$ are the known positive integers. The measurements are sampled non-uniformly. The subscript $i$ denotes the $i$th sensor and $L$ is the number of sensors.

**Assumption 1:** $w(t)$ and $v_i(k)$ are uncorrelated white noises with zero means and covariance matrices $E[w(t)w^T(t)] = Q_w$, $E[v_i(k)v_i^T(k)] = Q_{v_i}$, where $E$ denotes the mathematical expectation, the superscript $T$ denotes the transpose.

**Assumption 2:** The initial state $x(0)$ is uncorrelated with $w(t)$ and $v_i(k), i = 1, 2, \ldots, L$ and satisfies that

$$E\{x(0)\} = \mu_0, \quad E\{(x(0) - \mu_0)(x(0) - \mu_0)^T\} = P_0 \quad (3)$$

Our aim is to find the distributed fusion filter $\hat{x}_i(t | k)$ based on the received measurements $(y_i(k), \ldots, y_i(0))$, $i = 1, 2, \ldots, L$ measured by different sensors with asynchronous non-uniform sampling rates.

An example of time versus sensor map is depicted in Fig. 1, where the state updates uniformly at the highest rate and sensor 1 and 2 may sample non-uniformly at the lower sampling rates. The ratios of sampling rates between state and sensor are $S_1 : S_1 = 2$, and $S_2 : S_2 = 3$. The state data can be divided into blocks and the length of per block $N = m(n_1, n_2, \ldots, n_L)$ is the least common multiple of $n_i, i = 1, 2, \ldots, L$. The sensor $i$ should sample $p_i = N / n_i$ times per block. This means that the sampling of different sensors can be asynchronous. We refer the readers to [11-13] for more details.

![Figure 1. Sampling vs. sensor](image)

### III. DISTRIBUTED FUSION FILTER

#### A. System modeling

In the following, we will establish the non-augmented state models at each sensor by considering the system noises.

**Theorem 1:** Under Assumptions 1 and 2, the state space model at sensor $i$ can be set up as

$$\chi_i(k + 1) = \Phi_i \chi_i(k) + w_i(k) \quad (4)$$

$$y_i(k) = H_i \chi_i(k) + v_i(k), \quad i = 1, 2, \ldots, L \quad (5)$$

where

$$\Phi_i = \Phi^o \quad (6)$$

Also we have the following noise statistic information

$$Q_{w_i} = E[w_i(k)w_i^T(k)] = (1 / n_i) \sum_{m=0}^{n_i-1} \Phi^o \Gamma \sum_{m=0}^{n_i-1} \Phi^o \Gamma^T$$

$$Q_{v_i} = E[v_i(k)v_i^T(k)] = (1 / n_i) \sum_{m=0}^{n_i-1} \Phi^o \Gamma \sum_{m=0}^{n_i-1} \Phi^o \Gamma^T \Phi^o \Gamma$$

**Proof:** From (7), we have

$$\chi_i(k + 1) = \Phi^o \chi_i(k) + w_i(k)$$

where $\Phi_i, w_i(k)$ are defined by (6) and (8).

Remark 1: Differently from reference [10] where the state augmentation approach is used and the computational cost becomes more expensive as the length of per block $N$ increases, here the non-augmented state space models are established and the computational cost is obviously reduced.
Differently from references [11-13] where the established models are coarse since the system noise is ignored and the errors of established models become larger as the covariance matrix of the system noise $Q_\alpha$ increases, here the non-augmentation state space models are established by taking the system noise into account.

Remark 2: From (8)-(10), we see that the system noise $w(k)$ is colored noise which is correlated at the same and adjacent time instants, which implies $x(k)$ is also correlated with $w(k)$. Hence, the traditional Kalman filter is no longer applicable.

B. Local filters at the measurement sampling points

In the following, based on system (4)-(5), we will derive the local filters $\hat{x}(k | k)$ at the measurement sampling points and corresponding filtering error covariance matrices $P(k | k)$ by the innovation analysis approach.

**Theorem 2:** Under Assumptions 1 and 2, the local filter at the measurement sampling points of the $i$th sensor subsystem of system (4)-(5) is computed by

$$\hat{x}(k | k) = \hat{x}(k | k-1) + K_i(k)e_i(k)$$  \hspace{1cm} (11)

$$\hat{x}(k+1 | k) = \Phi\hat{x}(k | k-1) + L_i(k)e_i(k)$$  \hspace{1cm} (12)

$$e_i(k) = y_i(k) - H_i\hat{x}(k | k-1)$$  \hspace{1cm} (13)

$$K_i(k) = P(k | k-1)H_i^TQ_i^{-1}(k)$$  \hspace{1cm} (14)

$$L_i(k) = [\Phi P(k | k-1) + Q_i^{-1}]H_i^TQ_i^{-1}(k)$$  \hspace{1cm} (15)

$$Q_i(k) = H_iP(k | k-1)H_i^T + Q_i$$  \hspace{1cm} (16)

$$P(k | k) = P(k | k-1) - K_i(k)Q_i(k)K_i^T(k)$$  \hspace{1cm} (17)

$$P(k+1 | k) = (\Phi - L_i(k)H_i)P(k | k-1)(\Phi - L_i(k)H_i)^T + (\Phi - L_i(k)H_i)(Q_i^{-1})^T + Q_i^{-1}(\Phi - L_i(k)H_i)^T + Q_i^{-1} + L_i(k)Q_iL_i^T(k)$$  \hspace{1cm} (18)

where $e_i(k)$ is the innovation sequence with covariance matrix $Q_i^{-1}(k)$, $K_i(k)$ and $L_i(k)$ are the filtering and prediction gain matrices, $\hat{x}(k | k)$ and $\hat{x}(k | k-1)$ are the filter and predictor, $P(k | k)$ and $P(k | k-1)$ are the filtering and prediction error covariance matrices. The initial values are $\hat{x}(0 | 0) = \mu_0$ and $P(0 | 0) = P_0$.

Proof: By projection theory [18], we readily have (11)-(13), where the filtering gain matrix is defined as

$$K_i(k) = E[x(k)e_i^T(k)]Q_i^{-1}(k)$$  \hspace{1cm} (19)

Substituting (5) into (13), the innovation $e_i(k)$ can be rewritten as

$$e_i(k) = H_i\hat{x}(k | k-1) + v_i(k)$$  \hspace{1cm} (20)

Using (20), (19) and $x(k) \perp v_i(k)$, where the symbol $\perp$ denotes orthogonality, (14) is obtained. From (20), the innovation variance matrix (16) is obtained. The prediction gain matrix $L_i(k)$ is defined as

$$L_i(k) = E[x(k+1)\epsilon_i^T(k)]Q_i^{-1}(k)$$  \hspace{1cm} (21)

From (4) and (20), we have

$$E[w(k+1)\epsilon_i^T(k)] = \Phi E[w(k)\epsilon_i^T(k)]H_i^T + E[w(k)\hat{x}^T(k)]H_i^T$$  \hspace{1cm} (22)

Using (4) and (10), we have

$$E[w(k)\hat{x}^T(k)] = E[w(k)w^T(k-1)] - Q_i$$  \hspace{1cm} (23)

By substituting (23) and (22) into (21), (15) follows.

From (4), (11), (12) and (20), the filtering and prediction error equations are obtained as

$$\hat{x}(k | k) = x(k) - \hat{x}(k | k-1) - K_i(k)e_i(k)$$  \hspace{1cm} (24)

$$\hat{x}(k+1 | k) = \hat{x}(k+1 | k-1) + L_i(k)v_i(k)$$  \hspace{1cm} (25)

Using $P(k | k) = E[\hat{x}(k | k)\hat{x}^T(k | k)]$ and $\hat{x}(k | k) \perp e_i(k)$, we have the filtering error covariance matrix (17). From (25), $\hat{x}(k | k-1) \perp v_i(k)$ and $\hat{x}(k | k-1) \perp w_i(k)$, the prediction error covariance matrix is computed by

$$P(k+1 | k) = E[\hat{x}(k+1 | k)\hat{x}^T(k+1 | k)]$$  \hspace{1cm} (26)
By substituting (23) into (26), (18) follows.

C. Local filters at the state update points

Next, based on the local filter \( \hat{x}_i(k|k) \) at the measurement sampling points of system (4)-(5), we will derive the local filter \( \hat{x}(t|k) \) \( (n_k(k-1)+1 \leq t \leq n_k k) \) at the state update points of system (1)-(2).

**Theorem 3:** Under Assumptions 1 and 2, the local filter at the state update points of the \( i \)th sensor subsystem of system (1)-(2) is computed by

\[
\hat{x}_i(q|k) = n_i \Psi \hat{x}_i(k|k) - \Psi \sum_{j=0}^{n_k-1} \sum_{m=0}^{n_k-2-r} \Phi^m \Gamma_w \hat{w}_i(q+r, k), \quad q = n_i(k-1) + 1
\]  (27)

\[
\hat{x}_i(l|k) = \Phi^{l-r} \hat{x}_i(q|k) + \sum_{m=0}^{l-1-r} \Phi^m \Gamma_w \hat{w}_i(l-m-1|k),
\quad n_i(k-1) + 1 < l \leq n_k k
\]  (28)

\[
\hat{w}_i(q+r|k) = M_{w_i}(q+r|k) e_r(k), \quad r = 0, 1, \ldots, n_i-2
\]  (29)

\[
M_{w_i}(q+r|k) = \frac{1}{n_i} \sum_{m=0}^{n_k-2-r} Q_{w,} \Gamma^T (\Phi^m)^T H_i Q_{w_i}(k)
\]  (30)

where \( \Psi = \left( \sum_{m=0}^{n_k-2-r} \Phi^m \right)^{-1} \), \( \hat{x}_i(q|k) \) is the filter of the state \( x(q) \) \( (q = n_i(k-1)+1) \), \( \hat{x}_i(l|k) \) is the filter of the state \( x(l) \) \( (n_i(k-1)+1 < l \leq n_k k) \), \( \hat{w}_i(q+r|k) \), \( r = 0, 1, \ldots, n_i-2 \) is the input white noise filter, \( M_{w_i}(q+r|k) \) is the input white noise filtering gain matrix. The initial value \( \hat{x}_i(k|k) \) is computed by Theorem 2.

Proof: From (7), we have

\[
x(q) = x(n_i(k-1)+1) = n_i \Psi \hat{x}(k)
- \Psi \sum_{j=0}^{n_k-1} \sum_{m=0}^{n_k-2-r} \Phi^m \Gamma w(q+r), \quad q = n_i(k-1) + 1
\]  (31)

Taking projection of both sides of (31) onto the linear space spanned by \( L(y_i(k), y_i(k-1), \ldots, y_i(0)) \) yields (27). By iteration yields

\[
x(l) = \Phi^{l-q} x(q) + \sum_{j=0}^{l-1-q} \Phi^j \Gamma w(l-m-1)
\quad n_i(k-1) + 1 < l \leq n_k k
\]  (32)

Taking projection of both sides of (32) onto the linear space spanned by \( L(y_i(k), y_i(k-1), \ldots, y_i(0)) \), (28) is obtained.

Applying the projection theory \[18\], we have the input white noise filter

\[
\hat{w}_i(q+r|k) = \hat{w}_i(q+r|k-1) + M_{w_i}(q+r|k) e_r(k)
\quad q = n_i(k-1) + 1, \quad r = 0, 1, \ldots, n_i-2
\]  (33)

Using \( w(q+r) \perp L[y_i(k-1), y_i(k-2), \ldots, y_i(0)] \), we have \( \hat{w}_i(q+r|k-1) = 0 \). The input white noise filtering gain \( M_{w_i}(q+r|k) \) is defined as

\[
M_{w_i}(q+r|k) = E[w(q+r) e^{-T}(k)] Q_{w_i}(k)
\]  (34)

From (20) and \( w(q+r) \perp \hat{x}_i(k|k-1) \), we have

\[
E[w(q+r) e^{-T}(k)] = E[w(q+r) H_i (\hat{x}_i(k) - \hat{x}_i(k-1)) + v_i(k)]
\]

\[
= E[w(q+r) \hat{x}_i(k)] H_i = (1/n_i) \sum_{m=0}^{n_k-2-r} Q_w \Gamma^T (\Phi^m)^T H_i
\]  (35)

By substituting (35) into (34), (29) and (30) are obtained.

To apply the covariance intersection fusion estimation algorithm \[17\], we need the computation of covariance matrices of local filters, which is stated below.

**Theorem 4:** Under Assumptions 1 and 2, the local filtering error covariance matrix at the state update points of the \( i \)th sensor subsystem of system (1)-(2) is computed by

\[
P_i(q|k) = n_i^2 \Psi P_i(k|k) \Psi^T
- \Psi \left[ \sum_{m=0}^{n_k-2-r} \sum_{n=0}^{n_k-2-r} \Phi^m \Gamma Q_w \Gamma^T (\Phi^m)^T \Psi^T \right]
- \Psi \left[ \sum_{m=0}^{n_k-2-r} \sum_{n=0}^{n_k-2-r} \Phi^m \Gamma Q_w \Gamma^T (\Phi^m)^T (I - K_i(k) H_i)^T \Psi^T \right]
+ \Psi \left[ \sum_{m=0}^{n_k-2-r} \sum_{n=0}^{n_k-2-r} \Phi^m \Gamma P_w(q+r) \Gamma^T (\sum_{m=0}^{n_k-2-r} \Phi^m)^T \Psi^T \right]
- \Psi \left[ \sum_{j=0}^{l-1-q-r} \sum_{m=0}^{l-1-q-r} \Phi^m \Gamma M_{w_i}(q+j) Q_{w_i}(k) \right]
\times M_{w_i}(q+r|k) \Gamma^T (\sum_{m=0}^{n_k-2-r} \Phi^m)^T \Psi^T
\]  (36)

\[
P_i(l|k) = \Phi^{l-q} P_i(q|k) (\Phi^{q-l})^T
+ \Phi^{l-q} \Psi \left[ \left( (I - K_i(k) H_i) \sum_{m=0}^{n_k-1-q-r} \Phi^m \right) \Gamma Q_w \Gamma^T \right]
- \left( \sum_{m=0}^{n_k-1-q-r} \Phi^m \right) \Gamma P_w(q+r) \Gamma^T \Psi^T \right]
+ \left( \sum_{m=0}^{n_k-1-q-r} \Phi^m \Gamma Q_w \Gamma^T (\sum_{m=0}^{n_k-1-q-r} \Phi^m)^T (I - K_i(k) H_i)^T \Psi^T \right]
- \Psi \left( \sum_{m=0}^{n_k-1-q-r} \Phi^m \Gamma P_w(q+r) \Gamma^T (\sum_{m=0}^{n_k-1-q-r} \Phi^m)^T \Psi^T \right]
- \Psi \left( \sum_{j=0}^{l-1-q-r} \sum_{m=0}^{l-1-q-r} \Phi^m \Gamma M_{w_i}(q+j) Q_{w_i}(k) \right)
\times M_{w_i}(q+r|k) \Gamma^T (\sum_{m=0}^{n_k-2-r} \Phi^m)^T \Psi^T
\]  (37)

\[
P_{w_i}(q+r|k) = Q_{w_i} - M_{w_i}(q+r|k) Q_{w_i}(k) M_{w_i}(q+r|k)
\]  (38)
where \( P_i(q | k) \), \( q = n_i(k - 1) + 1 \) and \( P_i(l | k) \), \( n_i(k - 1) + 1 < l \leq n_k \) are the filtering error covariance matrices, \( P_{i+r}(q+r | k) \), \( r = 0, 1, \ldots, n_i - 2 \) are the estimation error covariance matrices of the input white noise \( w(q+r) \). The initial value \( P_i(k | k) \) is computed by Theorem 1.

Proof: Subtracting (27) from \( x(q) \), we have

\[
\tilde{x}_i(q | k) = n_i\Psi \tilde{x}_i(k | k) - \Psi \sum_{r=0}^{n_i-2} \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \Gamma \tilde{w}_i(q+r | k)
\]  

(39)

From (39), the filtering error covariance matrix \( P_i(q | k) \) is computed by

\[
P_i(q | k) = \mathbb{E}[(\tilde{x}_i(q | k))(\tilde{x}_i(q | k))^T]
= n_i \Psi P_i(k | k) \Psi^T
- n_i \Psi \left[ \sum_{r=0}^{n_i-2} \mathbb{E}[(\tilde{x}_i(q | k))(\tilde{w}_i(q+r | k))^T] \right] \Gamma \tilde{w}_i(q+r | k)
- n_i \Psi \left[ \sum_{r=0}^{n_i-2} \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \mathbb{E}[\tilde{w}_i(q+j | k) \tilde{x}_i(k | k)] \right] \Psi^T
+ \Psi \sum_{r=0}^{n_i-2} \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \Gamma \mathbb{E}[\tilde{w}_i(q+j | k)] \Psi^T
\times \tilde{w}_i^T(q+r | k) \Gamma \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \Psi^T
\]  

(40)

Subtracting (29) from \( w(q+r) \), \( r = 0, 1, \ldots, n_i - 2 \), we have the following input white noise filtering error equation

\[
\tilde{w}_i(q+r | k) = w(q+r) - M_{\omega_i} (q+r) \epsilon_i(k)
\]  

(41)

Then, (38) can be obtained readily by computing \( P_{i+r}(q+r | k) \) = \( \mathbb{E}[\tilde{w}_i(q+r) \tilde{w}_i^T(q+r)] \). Using (24), (20), (41), \( \tilde{x}_i(k | k) \perp \epsilon_i(k) \) and \( \tilde{x}_i(k | k-l) \perp w(q+r) \), we have

\[
\mathbb{E}[\tilde{x}_i(k | k) \tilde{w}_i^T(q+r | k)] = \mathbb{E}[(\tilde{x}_i(k | k) w^T(q+r))] = (I_{n_i} - K_i(k) H_i) \mathbb{E}[\tilde{x}_i(k) w^T(q+r)]
= (1/n_i) (I_{n_i} - K_i(k) H_i) \sum_{r=0}^{n_i-2} \Phi_m \Gamma Q_w
\]  

(42)

On the other hand, using Assumption 1, (34) and (41), we have

\[
\mathbb{E}[\tilde{w}_i(q+j | k) \tilde{w}_i^T(q+r | k)] = \begin{cases} P_{i+r}(q+j | k), & j = r \\ -M_{\omega_i} (q+j | k) Q_{\omega_i}(k) M_{\omega_i}^T (q+r | k), & j \neq r \end{cases}
\]  

(43)

Substituting (42), (43) into (40), (36) is obtained.

Subtracting (28) from (32), we have

\[
\tilde{x}_i(l | k) = \Phi^{l-q} \tilde{x}_i(q | k) + \sum_{m=0}^{l-q-1} \Phi_m \Gamma \tilde{w}_i(l-m-1 | k)
\]  

(44)

From (44), we have

\[
P_i(l | k) = \mathbb{E}[(\tilde{x}_i(l | k) \tilde{x}_i^T(l | k))]
= \Phi^{l-q} P_i(q | k) (\Phi^{l-q})^T
+ \Phi^{l-q} \sum_{r=0}^{l-q-1} \left[ \mathbb{E}[(\tilde{x}_i(q+r | k) \tilde{w}_i(l-m-1 | k) \Gamma (\Phi^{r})^T] \right]
+ \sum_{r=0}^{l-q-1} \left[ \Phi^{r} \Gamma \mathbb{E}[(\tilde{w}_i(l-m-1 | k) \tilde{x}_i(q+r | k))] (\Phi^{l-q})^T \right]
+ \mathbb{E}[\sum_{r=0}^{l-q-1} \Phi^{r} \Gamma \tilde{w}_i(l-m-1 | k) \tilde{w}_i^T(l-p-1 | k) \Gamma (\Phi^{p})^T]
\]  

(45)

From (39) and (41), we have

\[
\mathbb{E}[\tilde{x}_i(q | k) \tilde{w}_i^T(l-m-1 | k)] = n_i \Psi \mathbb{E}[\tilde{x}_i(k | k)] w^T(l-m-1)
- \Psi \sum_{r=0}^{n_i-2} \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \mathbb{E}[\tilde{w}_i(q+j | k)] \Psi^T
\times \tilde{w}_i^T(q+r | k) \Gamma \left( \sum_{m=0}^{n_i-2-r} \Phi_m \right) \Psi^T
\]  

(46)

Using (42), (43), (45) and (46), (37) is obtained.

D. Distributed fusion filter

Based on the local filters in Theorem 3 and the filtering error covariance matrices in Theorem 4, we have the following distributed fusion filter by applying covariance intersection fusion algorithm [17].

**Theorem 5:** Under Assumptions 1 and 2, the distributed weighted fusion filter \( \tilde{x}_i(t | k) \) at the state update points is given by

\[
\tilde{x}_i(t | k) = \sum_{i=1}^{L} A_i (t) \tilde{x}_i(t | k), \quad n_i(k-1) + 1 \leq t \leq n_k
\]  

(47)

where the local filters of the \( i \)th sensor \( \tilde{x}_i(t | k) = \tilde{x}_i(q | k) \) \( (q = n_i(k-1) + 1) \) and \( \tilde{x}_i(t | k) = \tilde{x}_i(l | k) \) \( (n_i(k-1) + 1 < l \leq n_k) \) are computed by Theorem 3. The fusion weights \( A_i(t) \) are computed by

\[
A_i(t) = \omega_i(t) \sum_{i=1}^{L} \omega_i(t) P_i^{-1}(t | k) - P_i^{-1}(t | k)
\]  

(48)

where \( \omega_i(t) \) are set as follows

\[
\omega_i(t) = \frac{\text{tr}(P_i^{-1}(t | k))}{\sum_{i=1}^{L} \text{tr}(P_i^{-1}(t | k))}, \quad 0 \leq \omega_i(t) \leq 1, \sum_{i=1}^{L} \omega_i(t) = 1
\]  

(49)

where \( P_i(t | k) = P_i(q | k) \), \( (q = n_i(k-1) + 1) \) and \( P_i(t | k) = P_i(l | k) \), \( (n_i(k-1) + 1 < l \leq n_k) \) are computed by Theorem 4.

Based on Theorems 1-4, the computational procedures of the distributed fusion filter at the state update points are as follows:

1) Using (11), we can obtain the local filters \( \tilde{x}_i(k | k) \), \( i = 1, 2, \ldots, L \) at measurement update points of system (4)-(5).
2) Using (27) and (28), we can obtain the local filters \( \tilde{x}_i(q | k) \) of the state \( x(q) (q = n_i(k-1) + 1) \) and \( \tilde{x}_i(l | k) \) of the
state $x(t) \ (n_i(k-1)+1 < t \leq n_i k)$ at state update points of system (1)–(2).

3) Using (36) and (37), we can obtain the local filtering error covariance matrices $P_i(q \mid k)$ and $P_i(l \mid k)$.

4) Using (48) and (47), we can obtain the fusion weights $A_i(t)$ and the distributed fusion filter $\hat{x}_o(t \mid k)$.

Above, all steps are implemented at each time step.

IV. SIMULATION RESEARCH

Consider a tracking system with 3 sensors

$$x(t+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} w(t)$$  \hspace{1cm} (50)$$

$$y_i(k) = [1 \ 0] x(k) + v_i(k), \ i = 1, 2, 3.$$  \hspace{1cm} (51)

where $T$ is the state update period. The state $x(t)=[s(t) \ \dot{s}(t)]^T$, where $s(t)$ and $\dot{s}(t)$ are the position and velocity of the target at time $tT$. The measurement noises $v_i(k), \ i = 1, 2, 3$ are independent Gaussian noises with zero means and variances $Q_{n_i}$ and are uncorrelated with $w(k)$. Our aim is to find the distributed fusion filter $\hat{x}_o(t \mid k)$ by covariance intersection algorithm. We take 150 sampling data. In simulation, we set $T = 0.1s, \ Q_v = 0.5, \ Q_n = 3.5, \ Q_{n_i} = 2.5, \ Q_{n_i} = 3, \ n_i = 2, \ n_2 = 3, \ n_3 = 1, \ x(0) = [0 \ 0]^T$ and $P_o = 0.1 l_z$.

Fig. 2 gives the tracking performances of the distributed fusion filter, where the solid curves denote the true values and the dashed ones denote the filters. Fig. 3 gives the comparison curves of MSEs simulated by 200-times Monte Carlo tests for all the local filters and the distributed fusion filter within the time $0 \sim 50$. From Fig. 3, we see that the accuracy of the fusion filter is better than that of all local filters.

![Figure 2. Distributed fusion filter](image)

![Figure 3. Comparison of MSEs of local filters and the distributed fusion filter](image)

Next, we make the comparison with reference [13], where the data received rate $\alpha = 1$. Fig. 4 shows the comparison of the MSEs curves of the two filters based on 200-times Monte Carlo tests. From Fig. 4, we see that the accuracy of our filter is significant better than that of [13]. Fig. 5 shows the comparison of the accumulated mean square errors of 150 sampling data over an average of 200 runs of Monte Carlo method (i.e., $\sum_{i=0}^{150} \left(1/200 \sum_{l=0}^{200} (\hat{x}_i^{(l)}(t) - x_i^{(l)}(t))^2 \right)$, the superscript $l$ denotes the $l$th simulation test). It can be observed that the accuracy of our fusion filter become better as $Q_{n_i}$ increases since the state models in [13] ignore the system noise. The simulation results verify the effectiveness of the proposed algorithm.
V. CONCLUSION

In this paper, the distributed fusion filtering problem for multi-rate multi-sensor non-uniform asynchronous sampling systems is investigated. The state is updated at the highest uniform rate and different sensors have different lower non-uniform measurement update rates. By considering the system noises, the non-augmented state models at each sensor are established. Based on the established state space models, local filters at the measurement sampling points and the local filters at the state update points are obtained by applying projection theory, respectively. Further, the distributed fusion filter is given by applying the covariance intersection fusion algorithm. The filtering error covariance matrices are derived to compute the fusion weights. Compared to reference [10], the computational cost is reduced since the non-augmentation approach is used. Compared to references [11-13], the estimation accuracy can be improved significantly since the system noise is considered.

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REFERENCES


