Abstract – Fusion of multiple fixes is an important in determining signal source location. Various fusion algorithms for multiple fixes and fix with bearing have been developed. However, the result from the commonly used fusion algorithm, which is normalized by error covariance matrix, is largely affected by the size and orientation of the respective error ellipse. In some applications, this could deduce an unreasonable estimation as the fused point may be largely biased from the centroid of these fixes. This paper first presents a unified fusion algorithm with weighted combination for multiple fixes, and then by taking various typical types of weights, including matrix or scale weights, we specify the formulations in these cases for different scenarios and purposes. Fusion of fix with bearing is also brought up as it is an important case to show the necessity for weighted fusion. Testing using a high-fidelity simulation system demonstrated that the algorithm well performed to realize different expectations.

Keywords: fusion algorithm; multiple fixes; error ellipse; triangulation of bearings.

I. INTRODUCTION

Multiple-sensor fusion is always important in determining signal source location and presenting the operator with an enhanced surveillance picture. In most applications, data from a sensor were in the form of bearing lines. Thus, one of the most essential fusion methods is the fusion algorithm for multiple bearings. Besides Stansfield’s algorithm [1,8,11,17], the pioneer work in this field, many algorithms have been developed, such as Orthogonal Vector (OV) algorithm, Total Least Square (TLS) algorithm for approximate linear least square estimations, and Maximum Likelihood approach for nonlinear least square estimation [1-10]. Gavish [2] also examined the performance analysis of several kinds of bearing-only target location algorithms. However, due to the difficulty of nonlinearity for both least square and ML estimations, all solutions, either in analytic formation, or in numeric computing formulation, are approximate solutions.

At present, a single sensor could produce fix directly. Under the new situation, fusion for multiple sensors is no longer just a triangulation of two bearing lines, it could include aggregation of multiple fixes, or bearing with fix. Furthermore, many application programs integrating multiple systems have been developed to improve the accuracy of estimation. In these programs, the fusion for multiple systems is naturally a problem of combination of multiple fixes, bearing with fix, or multiple bearings. Therefore, the combination of multiple fixes or fix with bearing line are critical in the applications of multiple sensors and multiple systems. Zhang, etc. [18] have proposed a method for fusion of multiple fixes. However, the output from this method, or from the commonly used joint Gaussian estimation, is largely affected by the sizes, shape, and orientations of error ellipses as shown in Fig.1, in which the fused fix is quite distant from the connecting line of two given fixes.

The size, shape, and orientation of error ellipse are determined by the sensor distribution and distance from sensors to target. In the fusion systems based on DOA (direction of arrival), the size and orientation of fixes reflect the certain information of the sensor distribution. In this case, the biased fused result may be reasonable. However, in the fusion systems based on TDOA (time difference of arrival), FDOA (frequency difference of arrival), or other measurements, the size and orientation of fixes are no longer directly related with sensor distribution. In this case, the fusion result, significantly biased from connecting line of two fixes, may not be most reasonable. Thus, it is important to produce a result closed to the connecting line of two fixes by reducing the affection of size and orientation in fusion method.

To reduce the affection of size and orientation of ellipse in fusion of fixes, one way is to adopt solely scale weights, and another way is to use weighted matrix. In this paper, we first present the formulation of a unified Weighted Combination for Multiple Fixes in Section 2, in which the weights could be chosen as scale, matrix or even a function. And then we specify the formulations for several typical cases of scale weight or matrix weights for different purposes in different applications in Section 3.

For choice of matrix weights, one of the natural choices is the error covariance, as in [18]. Another choice is covariance attached with confidence level. Suppose the fixes are obtained from different types of sensors systems, which each has a different confidence level by referring the sensor specifications, deployment, terrain condition, and even whether condition in operating, etc. In this case, the
confidences level for the sensor system should be considered in determination of weights.

For choice of scale weights, we know that the size and shape of error ellipse themselves represent a certain level of confidence. Thus, the parameters of sizes and shapes for error ellipses, like the dimension of area, the length of axes, the ranges of the sensors with targets, roundness of error ellipse, etc., could be considered in the choice of weights, as described in Section 3.

For clarity of description, in this paper a bearing line will be represented as \( B(x, y, \phi, \delta) \), where \((x, y)\) is the coordinate of the sensor position, \( \phi \) is the angle from true north to the bearing line in clockwise direction, and \( \delta \) is the angle variance for bearing \( \phi \). A fix will be denoted as \( P(x, y, a, b, \phi_k) \), where \((x, y)\) is the coordinate of the fix position, \( a \) and \( b \) are the major and minor axes of the error ellipses, respectively, and \( \phi \) is the angle from true north to the major axis of the error ellipse in clockwise direction.

This paper is organized as follows. First we present the general formulation for the weighted combination for multiple fixes in Section 2, and then further focus on some typical cases of weights for different scenarios in Section 3. As a special application, we restate the fusion algorithm of fix with bearing in [18] by affixing additional weights in Section 4. And Section 5 gives a series of examples by choosing different types of weights for different purposes, and makes some comparisons between these results. Finally we conclude the paper in Section 6.

II. WEIGHTED COMBINATION FOR MULTIPLE FIXES

This section will present the general fusion algorithm of weighted combination for multiple fixes.

Suppose there are \( n \) fixes with parameter sets, \( P_k(x_k, y_k, a_k, b_k, \phi_k) \) \((k = 1, 2, \ldots, n)\), the fusion of \( n \) fixes is to find an optimal estimation for the resultant fix, \( \hat{P}(\hat{x}, \hat{y}, \hat{a}, \hat{b}, \hat{\phi}) \), by optimizing the objective function, \( J \),

\[
J = \sum_{k=1}^{n} (x - x_k)^T W_k (x - x_k)
\]

where \( x = [x, y]^T \) and \( x_k = [x_k, y_k]^T \), and weight \( W_k \) could be chosen as matrix, scale or function. When \( W_k = w_k \) is scale, \( w_k > 0 \), and when \( W_k \) is matrix, it should be positive definite.

To solve the linear least square estimation, we could find the optimal estimation, \( \hat{x} = [x, y]^T \), by minimizing \( J \) as follows:

\[
\hat{x} = W^{-1} \sum_{k=1}^{n} w_k x_k
\]

where \( W = \sum_{k=1}^{n} w_k \), and the covariance, \( \Sigma \), for the estimation, \( \hat{x} \), could be deduced as

\[
\hat{\Sigma} = W^{-1} \left( \sum_{k=1}^{n} w_k \Sigma_k w_k^T \right) W^{-T} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}
\]

where \( W^{-1} = (W^{-1})^T \) and the covariance, \( \Sigma_k \), for the \( k \)th fix \( P_k \) is represented as

\[
\Sigma_k = \begin{bmatrix} \sin \phi_k & -\cos \phi_k \\ \cos \phi_k & \sin \phi_k \end{bmatrix} \begin{bmatrix} \sigma_k^2 \\ b_k^2 \end{bmatrix} \begin{bmatrix} \sin \phi_k & -\cos \phi_k \\ \cos \phi_k & \sin \phi_k \end{bmatrix}^T
\]

for \( k = 1, 2, \ldots, n \).

From (2b), the axes, \( \bar{a} \) and \( \bar{b} \), and orientation \( \bar{\phi} \) from true north of the error ellipse could be deduced as,

\[
\bar{a}^2 = \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_{12}^2} \right)
\]

\[
\bar{b}^2 = \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_{12}^2} \right)
\]

and

\[
\bar{\phi} = \frac{\pi}{2} - \frac{1}{2} \text{atan}2(2\sigma_{12}, \sigma_1^2 - \sigma_2^2)
\]

where the output of function \( \text{atan}2(.) \) is in the range of \(-\pi\) to \(\pi\), and therefore \( 0 \leq \bar{\phi} < \pi \).

III. TYPICAL CASES OF WEIGHTS

In practical applications, the size and shape of error ellipse themselves actually represent a certain level of confidence degree, thus the choice of weight should be related with some metric parameters, such as the dimension of area, the length of axes, roundness of error ellipse, etc. That is, the weight \( w_k \) should be a function of \( \sigma_k, b_k, \phi_k \).

1) Weights of Covariance

When the weights are taken as the inverse of covariance matrix in [18], that is \( W_k = \Sigma_k^{-1} \), the estimation in (2) will becomes

\[
\hat{x} = \Sigma_0 \left( \sum_{k=1}^{n} \Sigma_k^{-1} x_k \right)
\]

\[
\hat{\Sigma} = \Sigma_0 \left( \sum_{k=1}^{n} \Sigma_k^{-1} \Sigma_k \Sigma_k^{-T} \Sigma_0^{-T} = \Sigma_0 \right)
\]
\[ \Sigma_0^{-1} = \sum_{k=1}^{n} \Sigma_k^{-1} \]  

This is just the same formulation as in [18].

2) Weights of Weighted Covariance

Suppose fixes are attached with confidence levels by referring to sensor specifications, deployment, terrain condition, and whether condition in operating, etc. In this case, the confidence level for the sensor system should be considered in determining weights. One reasonable way is to multiply the confidence level \( c_k \) to respective covariance, that is \( w_k = c_k \Sigma_k^{-1} \). With this group of weights, the estimation in (2) will become

\[ \hat{x} = \Sigma_0 \left( \sum_{k=1}^{n} \frac{c_k}{w_k} \Sigma_k^{-1} \right) \quad (5a) \]

\[ \hat{\Sigma} = W^{-1} \left( \sum_{k=1}^{n} w_k \Sigma_k w_k^T \right) W^{-T} = \Sigma_0 \left( \sum_{k=1}^{n} c_k^2 \Sigma_k^{-1} \right) \Sigma_0^{-1} \quad (5b) \]

where

\[ \Sigma_0^{-1} = \sum_{k=1}^{n} c_k \Sigma_k^{-1} \]  

When all confidence degrees are 1s, this becomes the same formulation as in (4).

3) Weights of Confidence Degrees for Sensors

More simple and direct way is to consider the confidence degrees, \( c_k \), for sensors only by ignoring the measurement of error ellipses. In this case, \( w_k = c_k \), without considering the error ellipses for fixes.

4) Weights of Areas

As stated, the size and shape of error ellipse actually represent a certain level of confidence degree. Thus, the metric parameters of error ellipses, such as the dimension of area, the length of axes, roundness of error ellipse, etc., could be used in determination of weights. If the weights are taken as the reciprocal of the area size of the ellipse, that is \( w_k = 1/(a_k b_k) \), the estimation in (2) will be justified by an example in Section 5.

5) Weights of Axes

Also, the weights could be taken as the reciprocal of the length of axes, that is \( w_k = 1/(a_k + b_k) \), the estimation in (2) will also be tested with example in Section 5.

6) Weights of Shapes

In applications, it is assumed that roundness of ellipse also reflects a certain level of confidence degree, and the more roundness, the higher confidence degree. Considering the roundness and size of ellipses, the weights could be taken as \( w_k = b_k/(a_k b_k) \). With this type of weights, an example of the estimation in (2) will be tested in Section 5.

7) Weights of Equality

One of the simplest and most direct way is to choose the equal weights, that is \( w_k = 1/n, \) or \( w_k = 1 \). In this case, the estimation in (2) will be simplified as,

\[ \hat{x} = \frac{1}{n} \sum_{k=1}^{n} x_k \quad (6a) \]

and

\[ \hat{\Sigma} = \frac{1}{n^2} \sum_{k=1}^{n} \Sigma_k \quad (6b) \]

IV. FUSION OF FIX WITH BEARING

In fusion of fix and bearing, the fix is usually assigned higher weight as it provides more accurate information on the source location, and on the other hand the bearing is assigned with lower weight as it only gives the direction of the target.

Suppose weights to fix and bearing are denoted as \( w_f \) and \( w_b \), and usually \( w_b \leq w_f \), with these weights, and referring to Fig. 2, in which supposing the fix has the parameters \( P_f(x_f, y_f, a_f, b_f, \phi_f) \), the bearing line has parameter \( B_2(x_2, y_2, \phi_2) \), and the projection of \( P_f \) onto the bearing line is \( P_b(x_f, y_f, a_b, b_b, \phi_b) \), where \( a_b = b_b = r \) and \( r \) is the radius of the circle as described in [18]. Suppose the fused fix and ellipse as \( \hat{P}(\hat{x}, \hat{y}, \hat{a}, \hat{b}, \hat{\phi}) \), then the estimations for fused fix and ellipse can be represented as

\[ \hat{x} = \Sigma_0 \left( w_f \Sigma_f^{-1} x_f + w_b \Sigma_b^{-1} x_b \right) \quad (7a) \]

\[ \hat{\Sigma} = \Sigma_0 \left( w_f \Sigma_f^{-1} + w_b \Sigma_b^{-1} \right) \Sigma_0^{-1} \quad (7b) \]

where

\[ \Sigma_0^{-1} = w_f \Sigma_f^{-1} + w_b \Sigma_b^{-1} \]  

and the projection point \( x_b \) and the matrix corresponding to the circle \( \Sigma_b = \text{diag}(r^2, r^2) \) have been formulated in [18], and represented below.

\[ x_b = \frac{1}{2} \begin{bmatrix} x_f + x_2 \cos 2\phi_2 - \sin 2\phi_2 y_f + y_2 \\ -\cos 2\phi_2 + \sin 2\phi_2 \sin 2\phi_2 y_f + y_2 \end{bmatrix} \quad (8) \]

Also, the length of the straight segment, \( D \), from sensor position \( B_2(x_2, y_2) \) to the projection point \( P_b(x_b, y_b) \) could be computed from the formulation below,

\[ D = (x_f - x_2)\sin \phi_2 + (y_f - y_2)\cos \phi_2 \]  

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By using $D$, the radius $r$ could be computed as

$$r = D \tan \delta_2$$

(10)

**Example 2 – Weights of Weighted Covariance**

Fig. 4 shows the fusion of the two fixes using the weighted covariance as weight. In this example, we use two sets of different weights to demonstrate how the weights affect the fusion results. The first group of weights for weighted covariance is $\{0.5, 0.5\}$, and the second group of weights for weighted covariance is $\{0.9, 0.1\}$.

From these figures we can see that when two fixes are assigned with equal weights $\{0.5, 0.5\}$, the smaller sized fix dominates the result as shown in Fig. 4(a). However, this could be changed by assign different weights to fixes as shown in Fig. 4(b), in which when we assign larger weight 0.9 to the fix with larger error ellipse, and it will have larger affect to the fusion result.

**Example 3 – Weight of Areas**

When we choose scale weights, but also consider the uncertainty information from the dimensions of error ellipses, then the area sizes of error ellipses is one of suitable choices for scale weights. Fig. 5 shows the fusion result of the two fixes by using the areas of the ellipses as weight, in which the estimation of target position just lies on the connecting line of two fixes, and the result is clearly deviated to the fix with smaller area as it has higher confidence level.
Example 4 – Weight of Axes

Length of axes is another set of metric parameters of error ellipses, and therefore it also represents the uncertainty degree of ellipse. When we take the lengths of axes as the weights, the estimation of target position also lies on the connecting line of two fixes, but it is relatively closed to the middle point of the connecting line as shown in Fig. 6, as the lengths of axes, $a + b$, the first order quantity, has relatively slower changing rate than the size of area, $ab$, a second order quantity.

Example 5 – Weight of Shape

In applications, we assume that a round shaped ellipse has higher confidence degree than a narrow shaped ellipse under the condition of same size of area, as shown in Fig. 7. The roundness of an ellipse could be justified from ratio value $b/a$. By using the weights based on shapes of ellipses, $w_k = b_k/(a_k(a_k + b_k))$, we could obtain the fusion result as shown in Fig. 7, in which the more round-shaped ellipse has much higher weight than the narrow long shaped ellipse though they have the same area size, and therefore it has more affect to the fused result.

Example 6 – Weights of Equality

Equal weight is the simplest way for fusion of multiple fixes. In this case we do not care about the size and shape of each ellipse, and we just assign the equal weight, $w_k = 1/n$ or $w_k = 1$, to each fix. The fusion result in this case is displayed in Fig. 8, in which two fixes have the same effect to the final fusion result.

Example 7 – Weights of Confidence Degrees for Sensors

Regardless of sensor accuracy, sensor systems also have their reliable degree by considering the deployment, terrain, whether condition in operating, etc. Suppose sensor system for $P_1$ has 0.9 confidence degree, and sensor system for $P_2$ has 0.1 confidence degree. If we purely use the confidence degree as the scale weights to $P_1$ and $P_2$, the fusion result (red ellipse) is more deviated to the $P_1$ as displayed in Fig. 9, in which Fig. 9(a) is result from the weights of confidence degrees, and Fig. 9(b) is result from the equal weights.

This example shows that the confidence level of sensor systems is useful and effective in fusing of multiple fixes, if we know some sensor systems are more reliable, or more preferred, in practical applications.
Section 4 has described the fusion of fix with bearing. This example will demonstrate the results of different groups of weights for fix and bearing, to show the effect of weights in fusion of fix with bearing for more reasonable results.

Suppose the fix is denoted \( P_1=[27, 15, 7, 2, 75] \) and the bearing is denoted \( B_2=[1, 1, 50, 2] \), where the angles are degree measurement. When we assign equal weight, 1, to fix and bearing, the fused fix is deviated to the bearing as it has small angle covariance as in Fig. 10(a). When we assign lower weight 0.5 to bearing and higher weight 1 to fix, the fused fix is in the middle area of fix and bearing as in Fig. 10(b). What is more, when we further reduce the bearing weight to 0.1 and the fix weight is still to be 1, the fused result is quite closed to the fix as in Fig. 10(c). By adjusting the weights, we can obtain the estimation more depending on the fix, as it is more accurate and reliable than bearing.

VI. CONCLUSIONS

This paper presented a unified fusion algorithm with weighted combination for multiple fixes. Then we specified the formulations of several typical cases for different scenarios and purposes, by taking various types of weights, including matrix or scale weights. We especially investigated the method in fusion of fix with bearing, in which the weights to fix and bearing should be assigned with different values. Testing using a high-fidelity simulation system demonstrated that the algorithm well performed to realize different expectations.

References: