Abstract—In extended target tracking, targets potentially produce more than one measurement per time step. In recent random finite set (RFS) approaches, the set of measurements obtained from an extended target is modelled as a point process. In this paper, we expand on the RFS approach to extended target tracking by considering a hierarchical point process representation of multiple extended targets, more specifically a Poisson cluster process. This allows us to impose a geometric shape, in particular an ellipse, on each extended target. The set of target states, which are characterised by the kinematic variables and the shape parameters, represents the higher level (parent) process and the set of points on the boundary, from which measurements are generated, represents the lower level (daughter) process. We describe the PHD filter for multiple extended targets, whose extents vary in size, that estimates the shape parameters of the targets jointly with their positions and velocities. The main contribution of this paper is the practical implementation we propose, based on a particle-system representation for the targets’ shape and a Gaussian mixture formulation for the kinematic state per particle. The method is demonstrated on simulated data for multiple elliptical shaped extended targets.

Keywords: random finite sets, PHD filters, cluster processes, Gaussian mixture.

I. INTRODUCTION

The random finite set (RFS) approach to multi-target tracking has proven to be a useful method that allows the problem of estimating multiple dynamic targets in the presence of false measurements and detection uncertainty to be cast in a Bayesian filtering framework [1]. By treating the collection of individual targets as a set-valued state and the collection of observation measurements as a set-valued observation, this theoretically optimal approach is an elegant generalisation of the single-target Bayes filter. However the propagation of the posterior probability distribution in the multi-target Bayes filter is often computationally unattainable due to the high dimensionality of the target RFS. To overcome this intractability, the Probability Hypothesis Density (PHD) filter was introduced [1], which propagates the first-order moment, otherwise known as the intensity. The PHD filter is typically implemented using a particle representation applying sequential Monte Carlo (SMC) techniques [2], [3] or with a Gaussian mixture formulation [4]–[6] and has applications in radar [7], computer vision [8] and acoustics [9], to name but a few.

In most applications, including the PHD filter, it is assumed that each target produces at most one measurement per time-step. In extended target tracking, targets potentially produce more than one measurement per time-step. Multiple measurements per target per time-step gives rise to the possibility of estimating the target’s shape and size in addition to its position and velocity. An extension of the PHD filter to handle extended targets was presented in [10], recently implemented using a Gaussian mixture formulation [11], which involved representing the measurements generated from targets as a spatial point process [12], specifically a Poisson point process. That is, at each time-step, a Poisson distributed number of measurements are generated, spatially distributed around the target. Such a measurement model implies the target generated measurements resemble a cluster of points, rather than a geometrically structured arrangement.

A number of methods based on the RFS approach have been proposed for estimating the size and shape of a target [13], [14]. In [13], a direct application of the Gaussian mixture PHD filter for extended targets [11] is presented in which a method is proposed for calculating a likelihood function that relates a set of measurements, potentially, to a target whose structure resembles that of a rectangle or an ellipse. In [14], a partitioned state representation of an extended target is considered, which consists of linear (position & velocity) and non-linear (heading & target shape parameters) components. The PHD filter is applied for jointly estimating this target state and a set of measurement generating points defined on the boundary of each target’s geometrically structured shape (rectangle), and implemented using a Rao-Blackwellized particle filter as demonstrated for the Simulation Localisation and Mapping (SLAM) problem in [15]. Other proposed approaches for extended target size and shape estimation include a Random Hypersurface Model method [16] that assumes varying measurement sources which lie on scaled versions of shape boundaries, and a novel SMC approach [17] that considers target generated measurements bounded by a circular region which is estimated jointly with the target’s kinematic state.

In this paper we consider an alternative approach whereby multiple extended targets are modelled as a hierarchical point process as proposed in [18]. Hierarchical point processes, known as cluster point processes, have been previously exploited in the group target tracking problem [19], [20], the SLAM problem [21], and more recently in sensor registration.
In this paper, we specifically consider a Poisson cluster point process representation of multiple extended targets, which allows us to impose a geometric shaped extent on each target, where the centres of each target’s extent represent the higher level (parent) process on which a dynamic model can be specified, and the set of points on the boundary of each target’s extent represents the lower level (daughter) process from which the measurements are generated. In the PHD filter for this approach to extended target tracking, the kinematic states and the shape parametric variables are estimated jointly via the first-order moment.

The remainder of the paper is organised as follows: Sections II and III present the proposed methodology and implementation. The PHD recursion in Section II can easily be derived using a general chain rule and Gateaux differentials [23]–[25]. Section IV details a multiple extended target tracking example, whose extents are in shape and vary in size over time. The estimation results from the PHD filter implementation for this example using simulated data are then discussed, followed by concluding remarks in Section V.

II. EXTENDED TARGET TRACKING USING CLUSTER PROCESSES

In this work, we model multiple extended targets, whose extents have a geometrically shaped structure and vary in size, as a doubly-stochastic point process. In particular, we suppose the multiple extended targets have a Poisson cluster process representation, defined by \( \mathcal{X}_k = \{ (\mathbf{X}_{k,1}, \Xi_{k,1}), \ldots, (\mathbf{X}_{k,n}, \Xi_{k,n}) \} \) where for \( i = 1, \ldots, n \) the state \( \mathbf{X}_{k,i} = [x_{k,i}, s_{k,i}] \) consists of the random vector \( x_{k,i} \) defining the kinematic state and the random vector \( s_{k,i} \) defining the shape parameter variables for target \( i \). The RFS \( \Xi_{k,i} = \{ \xi_{k,1}, \ldots, \xi_{k,n} \} \) represents the points on the boundary of the extent for target \( i \). The parent process is represented by the RFS of target states \( \mathbf{X}_{k,i} \) with which there corresponds a daughter process, represented by the RFS \( \Xi_{k,i} \) conditioned on \( \mathbf{X}_{k,i} \), for \( i = 1, \ldots, n \).

In addition, we consider a measurement model, defined by the RFS \( \mathcal{Z}_k \), that takes into account detection uncertainty and false measurements (clutter) and is formed by the union of target generated measurements \( \Theta_k(\mathcal{X}_k) \) and clutter \( K_k \), i.e.

\[
\mathcal{Z}_k = K_k \cup \Theta_k(\mathcal{X}_k),
\]

where \( \lambda(z) \) is the intensity of the clutter process.

An integral part of extended target tracking with the PHD filter is the partitioning of the measurement set. A partition \( \pi \) is defined as a division of the RFS \( \mathcal{Z}_k \) into subsets, denoted as \( \varphi \), such that \( \bigcup_{\varphi \in \pi} \varphi = \mathcal{Z}_k \). We denote the set of all possible partitions of \( \mathcal{Z}_k \) as \( \Pi_{\mathcal{Z}_k} \). To assist the partitioning operation, instead of the false measurement model given by the probability density in (2), we model \( K_k \) as a Poisson cluster point process, the probability density for which is

\[
p_k(K_k) = \exp \left( - \int \lambda(z) \, dz \right) \prod_{\varphi \in \mathcal{Z}_k} \lambda(z),
\]

where \( \lambda(z) \) is the intensity of the clutter process.
where the predicted intensity of the shape parametric is
\[ M_{k|k-1}(s) = \int M_{k-1}(s') f_{k|k-1}(s | s') \, ds', \tag{6} \]
where the predicted intensity of the kinematic state is
\[ M_{k|k-1}(x | s) = \gamma_k(x | s) + \int p_S(x') f_{k|k-1}(x | x') M_{k-1}(x' | s) \, dx', \tag{7} \]
and \( \gamma_k(x | s) \) is the intensity for birth targets. The probability of survival \( p_S(x') \) is also assumed to be shape independent.

Given a new set of measurements \( Z_k \) and under the assumption that the daughter process can be approximated with a Poisson point process, the PHD update equation is
\[ M_k(s, x) = \frac{L_{Z_k}(s, x) M_{k|k-1}(s) M_{k|k-1}(x | s)}{\int L_{Z_k}(s, x) M_{k|k-1}(s) M_{k|k-1}(x | s) \, ds \, dx}, \tag{8} \]
where the joint pseudo-likelihood is
\[ L_{Z_k}(s, x) = e^{-\mu[1 | s,x]} + \sum_{\pi \in \Pi \pi_k} \omega_{\pi} \sum_{\varphi \in \varphi} \Lambda(\varphi) + M_{k|k-1}[\varphi], \tag{9} \]
with partition weight given as
\[ \omega_{\pi} = \frac{\prod_{\varphi \in \varphi} \left( \Lambda(\varphi) + M_{k|k-1}[\varphi] \right)}{\sum_{\pi' \in \Pi \pi_k} \prod_{\varphi' \in \varphi'} \left( \Lambda(\varphi') + M_{k|k-1}[\varphi'] \right)}, \tag{10} \]
given the following denotations
\[ \mu[1 | s, x] = \int \mu(\xi | s, x) \, d\xi, \]
\[ M_{k|k-1}[\varphi] = \int \varphi(s, x) M_{k|k-1}(x | s) \, dx, \]
\[ \varphi(s, x) = \exp \left( -\int \mu(\xi | s, x) \, d\xi \right) \]
\[ \times \prod_{z \in \varphi} \left( \int \mu(\xi | s, x) \, \lambda_2(z | \xi, x) \, d\xi \right), \tag{11} \]
for daughter process intensity \( \mu(\xi | s, x) \) and single-measurement likelihood \( \lambda_2(z | \xi, x) \). The clutter term is given by
\[ \Lambda(\varphi) = \lambda_1(u_{\varphi}) e^{-\lambda_2[1 | s, x]} \prod_{z \in \varphi} \lambda_2(z | u_{\varphi}), \tag{12} \]
for false measurements modelled as a Poisson cluster process whose probability density is defined by (3).

The likelihood \( L_{Z_k}(s, x) \) updates both the shape parametric \( s \) and kinematic state \( x \) and involves the summation over all possible partitions \( \Pi \pi_k \) of the measurement set. The number of partitions \( | \Pi \pi_k | \) for a measurement set of size \( | Z_k | = n \) equates to the \( n \)th Bell number \( (B_n = 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \ldots \) for \( n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots \) respectively). Consequently the operation of summing over all possible partitions in (9) is very computationally demanding. If a single partition, denoted as \( \pi \), is known the joint pseudo-likelihood reduces to
\[ L_{Z_k}(s, x) = e^{-\mu[1 | s,x]} + \sum_{\varphi \in \pi} \frac{\varphi(s, x)}{\Lambda(\varphi) + M_{k|k-1}[\varphi]}, \tag{13} \]
The study into appropriate partitioning methods is beyond the scope of this paper. However we assume, for the remainder of the paper, that such a partition \( \pi \) exists for each measurement set.

Finally, the presence of the lone exponential term in (13) can be accounted for by the Poisson point process approximation of the daughter process and the possible occurrence of an empty measurement set, attributing to missed detected targets. Hence this term can be regarded as the probability of missed detection, which for a daughter process with Poisson rate greater than 3 is less than 0.05, and can be interpreted in the following way, that the more boundary points on a target’s extent, the more measurements are likely to be generated resulting in a low probability of missed detection.

### III. IMPLEMENTATION

We implement the PHD filter for the multiple extended target model, described in Section II, using a Dirac mixture model for the intensity of the target shape. Each component of this mixture model has associated with it a Gaussian mixture formulation for the intensity of the kinematic state conditioned on that particular shape component. At each iteration of the filter, we start with the following prior intensities
\[ M_{k-1}(s) = \sum_{i=1}^{N_{k-1}} \omega_{k-1}^{(i)} \delta(s - s_{k-1}^{(i)}), \tag{14} \]
\[ M_{k-1}(x | s_{k-1}^{(i)}) = \sum_{j=1}^{J_{k-1}^{(i)}} \omega_{k-1}^{(j|i)} \mathcal{N}(x | s_{k-1}^{(i)}; m_{k-1}^{(j|i)}, P_{k-1}^{(j|i)}). \tag{15} \]

The notation \( \mathcal{N}(x; m, P) \) is used to denote a Gaussian distribution with mean vector \( m \) and covariance matrix \( P \), and \( \delta(s) \) denotes the Dirac delta distribution centred at \( s \). The mixture in (14) defines a particle representation of \( M_{k-1}(s) \), which can be expressed as the set of weighted particles \( \{\omega_{k-1}^{(i)}, s_{k-1}^{(i)}\}_{i=1}^{N_{k-1}} \).

#### A. Gaussian mixture formulation

Following the derivation of the Gaussian mixture PHD filter for single measurement targets in [4] and the single-group PHD filter in [20], a Gaussian mixture formulation can be attained for the predicted intensity of the kinematic state in (7) and the updated intensity of the kinematic state defined by
\[ M_k(x | s_{k-1}^{(i)}) = L_{Z_k}(s_{k}^{(i)}, x) M_{k|k-1}(x | s_{k-1}^{(i)}), \tag{16} \]
where \( L_{Z_k}(s_{k}^{(i)}, x) \) is the joint pseudo-likelihood given in (9) for each particle \( s_{k}^{(i)} \).
For the predicted intensity $M_{k|k-1}^{(i)}(x)$, we assume a linear Gaussian dynamic model given by

$$f_{k|k-1}(x | x') = \mathcal{N}(x; Fx', Q),$$  \hspace{1cm} (17)$$
where $F$ is the state transition matrix and $Q$ is the process noise covariance matrix. In addition suppose the probability of survival is state independent, i.e. $p_S(x') = p_S$, and the intensity for the birth targets is a Gaussian mixture of the form

$$\gamma_k(x | s_k^{(i)}) = \sum_{j=1}^{J_{\gamma,k}^{(i)}} \omega_{\gamma,k}^{(j)} \mathcal{N}(x | s_k^{(i)}; m_{\gamma,k}^{(j)}, P_{\gamma,k}^{(j)}),$$  \hspace{1cm} (18)$$
where $J_{\gamma,k}^{(i)}$, $\omega_{\gamma,k}^{(j)}$, $m_{\gamma,k}^{(j)}$ and $P_{\gamma,k}^{(j)}$ for $j = 1, \ldots, J_{\gamma,k}^{(i)}$ are given model parameters that determine the shape of the birth intensity for a particular particle $i$. The prediction for the kinematic state is then the same as that for the standard Gaussian mixture PHD filter

$$M_{k|k-1}(x | s_k^{(i)}) = \sum_{j=1}^{J_{\gamma,k}^{(i)}} \omega_{\gamma,k}^{(j)} \mathcal{N}(x | s_k^{(i)}; m_{\gamma,k}^{(j)}, P_{\gamma,k}^{(j)}),$$  \hspace{1cm} (19)$$
where for birth targets we have $\omega_{\gamma,k}^{(j)} = \omega_{\gamma,k}^{(j)}$, $m_{\gamma,k}^{(j)} = m_{\gamma,k}^{(j)}$, $P_{\gamma,k}^{(j)} = P_{\gamma,k}^{(j)}$, for persistent targets

$$\omega_{\gamma,k}^{(j)} = p_S \omega_{\gamma,k}^{(j)}, \quad m_{\gamma,k}^{(j)} = Fm_{\gamma,k}^{(j)} - \delta_{\gamma,k}^{(j)}, \quad P_{\gamma,k}^{(j)} = FP_{\gamma,k}^{(j)}F^T + Q,$$  \hspace{1cm} (20)$$
and $J_{\gamma,k}^{(i)} = J_{\gamma,k}^{(i)}$. For the updated intensity $M_k(x | s_k^{(i)})$ given in (16), we assume a separable linear Gaussian single-measurement likelihood given by

$$l_{\varphi}(\xi, x) = \mathcal{N}(z; H\xi, R_1) \times \mathcal{N}(z; \tilde{H}x, R_2),$$  \hspace{1cm} (21)$$
where $\mathcal{N}(z; H\xi, R_1)$ is the likelihood that the measurement $z$ is related to the point $\xi$ on the boundary of the target’s shape, with projection matrix $H$ and observation noise covariance matrix $R_1$, and $\mathcal{N}(z; \tilde{H}x, R_2)$ is the likelihood that the measurement $z$ relates to the state vector $x$, with projection matrix $\tilde{H}$ and observation noise covariance matrix $R_2$. In addition suppose the intensity of the daughter process is a Gaussian mixture of the form

$$\mu(\xi | s_k^{(i)}) = \sum_{\ell=1}^{J_\ell^{(i)}} v_{\ell}^{(i)} \mathcal{N}(\xi | s_k^{(i)}; m_{\xi,k}^{(i)}, P_{\xi,k}^{(i)}),$$  \hspace{1cm} (22)$$
where the mean and covariance are given by

$$m_{\xi,k}^{(i)} = m_{\xi}^{(i)} + P_{\xi,12}^{(i)} (P_{\xi,k}^{(i)})^{-1} (x - m_{\xi,k}^{(i)}) ,$$

$$P_{\xi,k}^{(i)} = P_{\xi,11}^{(i)} - P_{\xi,12}^{(i)} (P_{\xi,k}^{(i)})^{-1} P_{\xi,12}^{(i)} ,$$  \hspace{1cm} (23)$$
The Gaussian components in (22) are conditional Gaussian realisations from joint Gaussians with mean $[m_{\xi}^{(i)}, m_{\xi,k}^{(i)}]^T$ and covariance given by the block matrix

$$\begin{bmatrix}
P_{\xi,11}^{(i)} & P_{\xi,12}^{(i)} \\
(P_{\xi,12}^{(i)})^T & P_{\xi,k}^{(i)}
\end{bmatrix},$$  \hspace{1cm} (24)$$
for $j = 1, \ldots, J_{\gamma,k}^{(i)}$ and $\ell = 1, \ldots, J_\ell^{(i)}$, where $m_{\xi}^{(i)}$, $m_{\xi,k}^{(i)}$ are the predicted means and covariances from (19), $P_{\xi,k}^{(i)}$ are the predicted means and covariances from the corresponding marginal Gaussian for the daughter process, and $P_{\xi,12}^{(i)}$ are the covariances that describe the relationship between parent and daughter.

As a consequence of the Gaussian mixture formulation for the daughter intensity, in the joint pseudo-likelihood given in (13), we have $e^{-\mu(\xi | s_k^{(i)}; x)} = e^{-\mathcal{L}}$, denoting the expected number of boundary points as $\alpha = \sum_{\ell=1}^{\xi} v_{\ell}^{(i)}$, and in $T_{\varphi}(s, x)$ we have

$$\prod_{\varphi \in \varphi} \mu(\varphi | s_k^{(i)}; x) = \prod_{\varphi \in \varphi} \mathcal{N}(z; \tilde{H}x, R_2) \times \sum_{\ell=1}^{J_\ell^{(i)}} v_{\ell}^{(i)} \mathcal{N}(\varphi | s_k^{(i)}; \mathcal{H}m_{\xi,k}^{(i)}, \mathcal{R}_{\ell}^{(i)}) ,$$  \hspace{1cm} (25)$$
where, for brevity, we denote $\mathcal{R}_{\ell}^{(i)} = H \mathcal{P}_{\xi,k}^{(i)} H^T + R_1$. This product of Gaussian mixtures can be expanded for $\bar{\ell} = 1, \ldots, J_\ell^{(i)}$ and $\bar{\iota} = 1, \ldots, |\varphi|$ to give the following sum of Gaussian products

$$\sum_{(j_1, \ldots, j_{\bar{\ell}}) \in \varphi} \left( \prod_{\bar{\iota}=1}^{\bar{\ell}} v_{\bar{\iota}}^{(j_{\bar{\iota}})} \mathcal{N}(z_1^{(j_{\bar{\iota}})}; \tilde{H}x, R_2) \right) \times \mathcal{N}(z_1^{(j_{\bar{\iota}})}; \mathcal{H}m_{\xi,k}^{(i)}, \mathcal{R}_{\ell}^{(i)}) \right) .$$  \hspace{1cm} (26)$$
Now for further brevity, we denote $j_{\bar{\iota}} = (j_1, \ldots, j_{\bar{\iota}})$, then the measurement update for the kinematic state is

$$M_k(x | s_k^{(i)}) = e^{-\alpha} M_{k|k-1}(x | s_k^{(i)}) +$$

$$\sum_{\varphi \in \varphi} \sum_{j_{\bar{\iota}} = 1}^{J_\ell^{(i)}} \sum_{j_{\bar{\iota}} = 1}^{J_\ell^{(i)}} \omega^{(j_{\bar{\iota}})} \mathcal{N}(x | s_k^{(i)}; m_{k,j_{\bar{\iota}}}, P_{k,j_{\bar{\iota}}}),$$

where for $1 \leq \bar{\iota} \leq |\varphi|$ the means $m_{k,j_{\bar{\iota}}}$ and covariances $P_{k,j_{\bar{\iota}}}$ in the Gaussian components are

$$m_{k,j_{\bar{\iota}}} = m_{k,j_{\bar{\iota}}} + K_{k,j_{\bar{\iota}}} (z_1 - \tilde{H}m_{k,j_{\bar{\iota}}}),$$

$$P_{k,j_{\bar{\iota}}} = (I - K_{k,j_{\bar{\iota}}}) \tilde{H} P_{k,j_{\bar{\iota}}} \tilde{H}^T ,$$  \hspace{1cm} (28)$$
and \( K_{k}^{(ji)\{ji\}} = \tilde{P}_{k}^{(ji)\{ji\}} \tilde{H}^{T} \left( S_{2}^{(ji)\{ji\}} \right)^{-1} \). The weights of the Gaussian components are

\[
\omega_{\varphi, k}^{(ji)\{ji\}} = \omega_{k}^{(ji)\{ji\}} \exp \left\{ -\frac{1}{2} \mathbf{z}_{i}^{\top} S_{k}^{(ji)\{ji\}}^{-1} \mathbf{z}_{i} \right\},
\]

where

\[
L_{j_{k}|i}^{(ji)\{ji\}}(\mathbf{s}_{k}^{(i)}) = \sum_{i=1}^{M} \omega_{\varphi, k}^{(ji)\{ji\}}(\mathbf{s}_{k}^{(i)}) 
\]

in which we have

\[
q_{1}^{(ji)\{ji\}}(\mathbf{z}_{i} | \mathbf{s}_{k}^{(i)}) = \mathcal{N}(\mathbf{z}_{i} | \mathbf{s}_{k}^{(i)}; \mathbf{b}_{1}^{(ji)\{ji\}}, \mathbf{S}_{1}^{(ji)\{ji\}}),
\]

\[
q_{2}^{(ji)\{ji\}}(\mathbf{z}_{i} | \mathbf{s}_{k}^{(i)}) = \mathcal{N}(\mathbf{z}_{i} | \mathbf{s}_{k}^{(i)}; \tilde{H}\mathbf{m}_{k}^{(ji)\{ji\}}, \mathbf{S}_{2}^{(ji)\{ji\}}),
\]

and, denoting \( \mathcal{A}^{(ji)\{ji\}} = \mathcal{H} \psi_{\epsilon, 12} \) is

\[
\mathcal{A}^{(ji)\{ji\}} = \mathcal{H} \mathcal{P}^{(ji)\{ji\}} \left( \mathcal{P}^{(ji)\{ji\}} \right)^{-1},
\]

\[
b_{1}^{(ji)\{ji\}} = \tilde{H} \mathbf{m}_{k}^{(ji)\{ji\}} + \mathcal{A}^{(ji)\{ji\}} (\mathbf{m}_{k}^{(ji)\{ji\}}),
\]

\[
\mathbf{S}_{1}^{(ji)\{ji\}} = \mathcal{A}^{(ji)\{ji\}} \mathbf{S}_{1}^{(ji)\{ji\}} \mathcal{A}^{(ji)\{ji\}}^{T} + \mathbf{R}^{(ji)\{ji\}},
\]

Furthermore, for \( 1 \leq i \leq |\varphi| \)

\[
\tilde{m}_{k}^{(ji)\{ji\}} = \eta_{k}^{(ji)\{ji\}} + \mathcal{K}_{k}^{(ji)\{ji\}} (\mathbf{z}_{i} - \mathbf{b}_{1}^{(ji)\{ji\}}),
\]

\[
\tilde{P}_{k}^{(ji)\{ji\}} = \left( \mathbf{I} - \mathcal{K}_{k}^{(ji)\{ji\}} \mathcal{A}^{(ji)\{ji\}} \right) \mathbf{S}_{k}^{(ji)\{ji\}},
\]

where

\[
\eta_{k}^{(ji)\{ji\}} = \left\{ \begin{array}{ll}
\mathbf{m}_{k-1}^{(ji)} & \text{for } i = 1, \\
\mathbf{m}_{k-1}^{(ji)\{i\}} & \text{for } 1 < i \leq |\varphi|,
\end{array} \right.
\]

\[
\mathbf{S}_{k}^{(ji)\{ji\}} = \left\{ \begin{array}{ll}
\mathbf{P}_{k-1}^{(ji)\{ji\}} & \text{for } i = 1, \\
\mathbf{P}_{k-1}^{(ji)\{i\}} & \text{for } 1 < i \leq |\varphi|.
\end{array} \right.
\]

The resulting mixture in (27) has \( j_{k}^{(ji)\{ji\}}\{ji\} \times |\varphi| \times \mathcal{A}^{(ji)\{ji\}} \) components and left unchecked the size of the parent process mixture would grow at an approximately exponential rate with every time step. Many of these come from low-likelihood measurement associations and contribute little to the updated intensity in (27). There are a couple of Gaussian mixture reduction procedures we employ to then eliminate and merge such low-weighted components. The first is a measurement gating procedure as described in [27] which we apply to determine how many boundary points \( \xi \) are related to measurements \( \mathbf{z} \in \varphi \) for each \( \varphi \in \pi \), hence reducing the number of components in the mixture within the product given in (25). The other procedure is the pruning and merging one presented in [4]. Applying both procedures help maintain a more manageable approximation of the kinematic state intensity.

### B. Particle representation

To implement the Gaussian mixture formulation in Section III-A, we sample \( N_{k-1} \) particles from the Markov transition density for the particle representation of \( M_{k}(\mathbf{s}) \), i.e. for \( i = 1, \ldots, N_{k-1} \), sample \( \mathbf{s}_{k}^{(i)} = \pi_{k-1}(\mathbf{s} | \mathbf{s}_{k-1}^{(i)}) \). The weights of the Dirac mixture can be updated then as follows

\[
\omega_{k}^{(i)} = \frac{L_{Z_{k}}(\mathbf{s}_{k}^{(i)})}{\sum_{i=1}^{N_{k-1}} L_{Z_{k}}(\mathbf{s}_{k}^{(i)})},
\]

where \( L_{Z_{k}}(\mathbf{s}_{k}^{(i)}) \) can be determined from the updated intensity for the process

\[
L_{Z_{k}}(\mathbf{s}_{k}^{(i)}) = e^{-\alpha} \sum_{j=1}^{j_{k}^{(i)\{i\}}} \omega_{k}^{(i)\{j\}} + \sum_{j=1}^{j_{k}^{(i)\{i\}}} \sum_{j_{k}^{(i)\{j\}}} \omega_{k}^{(i)\{j\}},
\]

such that \( \omega_{k}^{(i)\{j\}} \) is as defined in (29). A resampling procedure is then applied, which makes \( N_{k} \) copies of particles \( \{ \omega_{k}^{(i)\{j\}} \} \}_{j=1}^{N_{k}} \) and eliminates low-weighted particles to give \( \omega_{k, s_{k}}^{(i)} \}_{i=1}^{N_{k}} \).

### IV. SIMULATED EXAMPLE

The methodology and implementation detailed in Sections II & III apply to extended targets with any structural shape in general. In this section we present a multiple extended target tracking example in which the extents are, specifically, elliptical in shape, that the proposed implementation has been tested on, using simulated data. The generated target trajectories are shown in Fig. 1 along with the elliptical shapes, parameterized by the major and minor axes, which gradually reduce in size over 100 iterations, for each target. The target trajectories are also shown plotted against time, along with the simulated measurements, in Fig. 2.

In the following subsection we first provide precise details of the simulation set-up before presenting the simulation results in Section IV-B.

#### A. Set-up

The target state \( \mathbf{x}_{k} \) consists of the kinematic state variable

\[
\mathbf{x}_{k} = \left[ x_{k,1}, x_{k,2}, x_{k,3}, x_{k,4} \right]^{T},
\]

where \( \left[ x_{k,1}, x_{k,3} \right]^{T} \) is the position vector (centre of the ellipse) and \( \left[ x_{k,2}, x_{k,4} \right]^{T} \) is the velocity vector at time-step \( k \), and shape parametric variable

\[
\mathbf{s}_{k} = \left[ s_{k,1}, s_{k,2}, s_{k,3}, s_{k,4} \right]^{T},
\]

where \( s_{k,1} \) is the magnitude of the major axis and \( s_{k,3} \) is the magnitude of the minor axis of the ellipse, while \( s_{k,2} \) & \( s_{k,4} \) are the rates of change in the major and minor axes’ magnitudes respectively.

Targets follow the linear Gaussian dynamic model (17) with

\[
\mathbf{F} = \begin{bmatrix} F_{k} & 0_{2} \\ 0_{2} & F_{k} \end{bmatrix}, \quad Q = \sigma_{z}^{2} \Gamma_{k} \Gamma_{k}^{T},
\]

where \( F_{k} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).
Figure 1: The multiple target trajectories (line) and size varying elliptical extents. Starting positions are indicated by the blue ‘filled diamonds’ at times (a) \( k = 1 \), (b) \( k = 26 \), (c) \( k = 51 \) and terminal positions are indicated by the blue ‘filled squares’ at times (a) \( k = 100 \), (b) \( k = 125 \), (c) \( k = 150 \).

Figure 2: Measurements (green ‘×’) and target trajectories (line).

such that

\[
F_k = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \quad \Gamma_k = \begin{bmatrix} \tau^2 / 2 & \tau & 0 & 0 \\ \tau & 0 & \tau^2 / 2 & \tau \end{bmatrix},
\]

where \( O_2 \) denotes the \( 2 \times 2 \) zero matrix, \( \tau = 1 \)s is the sampling period and \( \sigma_x = \sqrt{2} \) \((\text{m/s}^2)\) is the standard deviation of the process noise. The probability of survival is \( p_S = 0.9 \).

The point measurements \( \mathbf{z} = [z_1, z_2]^T \in Z_k \) relate to points \( \xi \) on the ellipse according to the likelihood \( \mathcal{N}(\mathbf{z}; \mathbf{H}\xi, \mathbf{R}_1) \) in (21) with \( \mathbf{H} = \mathbf{I}_2 \) and \( \mathbf{R}_1 = \sigma_1^2 \mathbf{I}_2 \) where \( \sigma_1 = \sqrt{5} \)m. They relate to the target state \( \mathbf{x} \) according to the likelihood \( \mathcal{N}(\mathbf{z}; \tilde{\mathbf{H}}\mathbf{x}, \mathbf{R}_2) \) in (21) with

\[
\tilde{\mathbf{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

and \( \mathbf{R}_2 = \sigma_2^2 \mathbf{I}_2 \) where \( \sigma_2 = \sqrt{50} \)m. The average number of observed points on the ellipse is \( \alpha = 5 \). The clutter term in the PHD update is given by (12) with \( \lambda_1(\mathbf{u}_\varphi) = \kappa_1 U(\mathbf{u}_\varphi) \) and \( \lambda_2(\mathbf{z} | \mathbf{u}_\varphi) = \kappa_2 \mathcal{N}(\mathbf{z}; \mathbf{u}_\varphi, \mathbf{R}_s) \) where \( U(\mathbf{u}_\varphi) \) is the uniform density over the observation region, \( \kappa_1 = 20 \) is the average number of false measurement clusters over the region, \( \kappa_2 = 1 \) is the average number of measurements per cluster and \( \mathbf{R}_s = 5 \times \mathbf{I}_2 \).

The filter is initialised with the birth intensity \( \gamma_k(\mathbf{x} | \mathbf{s}_k^{(i)}) \) at time-step \( k = 1 \). Birth targets are measurement driven, a technique applied for the PHD filter in [28]. That is, in (18), for measurements in each cluster subset \( \varphi \in \pi \) and \( j = 1, \ldots, J_{\gamma,k}^{(i)} \), where \( J_{\gamma,k}^{(i)} = |\pi| \), we have

\[
\mathbf{m}_{\gamma,k}^{(i)} = \begin{bmatrix} \frac{1}{|\varphi|} \sum_{z \in \varphi} z_1 & 0 \\ 0 & \frac{1}{|\varphi|} \sum_{z \in \varphi} z_2 & 0 \end{bmatrix},
\]

with covariance \( \mathbf{P}_{\gamma,k}^{(i)} = \sigma_2^2 \mathbf{I}_4 \) and weight \( \omega_{\gamma,k}^{(i)} = 0.1 \), for each particle \( i \).

The sampling distribution for the particle representation of a target’s shape is a linear truncated Gaussian. That is

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Figure 3: Position estimates (blue ‘circles’), uncertainty (grey shaded area) and target trajectories (line).

Figure 4: Shape estimates (blue ‘circles’), uncertainty (grey shaded area) and true shape variation line.

for persistent targets, we sample \( \tilde{s}_k^{(i)} \sim f_{k|k-1}(s | s_k^{(i-1)} = N(s; \Psi s_k^{(i-1)}, \tilde{Q}) \) such that \( \tilde{s}_k^{(i)} \) lies within the interval \((a, b)\) where

\[
a = [19 \ -0.2 \ 14 \ -0.1]^T, \quad b = [32 \ 0 \ 21 \ 0]^T,
\]

\(\Psi\) is as defined in (39) and

\[
\tilde{Q} = \Gamma_k \Sigma_{\gamma} \Gamma_k^T, \quad \Sigma_{\gamma} = \text{diag}([\sigma_{s,1}^2, \sigma_{s,2}^2, \sigma_{s,3}^2, \sigma_{s,4}^2]),
\]

for \(\sigma_{s,1} = \sqrt{2}\text{m}, \sigma_{s,2} = \sqrt{0.2}\text{m}, \sigma_{s,3} = \sqrt{\text{Th}}, \sigma_{s,4} = \sqrt{0.\text{Th}}\) and \(\Gamma_k\) as defined in (40). For new targets, we sample \(c_k^{(i)} = \begin{bmatrix} c_{k,1}^{(i)} & c_{k,2}^{(i)} \end{bmatrix}^T \sim \mathcal{N}(c; b_{\text{new}}, \Sigma_{\text{new}})\) such that \(c_k^{(i)}\) lies within the interval \((a_{\text{new}}, b_{\text{new}})\), where \(a_{\text{new}} = [19 \ 14]^T\), \(b_{\text{new}} = [31 \ 21]^T\) & \(\Sigma_{\text{new}} = \text{diag}([\sigma_{s,1}^2, \sigma_{s,2}^2])\), so that we have \(s_k^{(i)} = \begin{bmatrix} c_{k,1}^{(i)} \ 0 \ c_{k,2}^{(i)} \ 0 \end{bmatrix}^T\).

Finally, the daughter process for each particle \(i\) and kinematic state \(j\) has the conditional Gaussian mixture form given in (22) with \(J_{\xi}^{(ij)} = \alpha, \nu_{\xi}^{(ij)} = 1\),

\[
\begin{align*}
\mathbf{m}_{\xi}^{(ij)} &= \tilde{H}m_{k|k-1}^{(i)} + v_{k|k-1}^{(ij)} , \quad \mathbf{P}_{\xi,11} = \mathbf{R}_1 , \\
\mathbf{P}_{\xi,12}^{(ij)} &= \beta \sigma_1 \begin{bmatrix} \sqrt{\mathbf{P}_{k|k-1}^{(ij)}(1,1)} & 0 & 0 & 0 \\
0 & 0 & \sqrt{\mathbf{P}_{k|k-1}^{(ij)}(3,3)} & 0 \end{bmatrix}.
\end{align*}
\]

for all \(\ell = 1, \ldots, J_{\phi}^{(ij)}\), where

\[
t_{k|k-1}^{(ij)} = \begin{bmatrix} \hat{s}_k^{(i)} \cos(\theta_k) \cos(\phi_{k|k-1}^{(ij)}) + \hat{s}_k^{(i)} \sin(\theta_k) \sin(\phi_{k|k-1}^{(ij)}) \\
\hat{s}_k^{(i)} \sin(\theta_k) \sin(\phi_{k|k-1}^{(ij)}) - \hat{s}_k^{(i)} \sin(\theta_k) \cos(\phi_{k|k-1}^{(ij)}) \end{bmatrix},
\]

given \(\theta_k = 2\pi \ell / \alpha, \phi_{k|k-1}^{(ij)} = \arctan(m_{k|k-1,n}^{(ij)}/m_{k|k-1,2}^{(ij)})\), denoting \(m_{k|k-1,n}^{(ij)}\) as the \(n^{th}\) element of mean vector \(\mathbf{m}_{k|k-1}^{(ij)}\), and \(\mathbf{P}_{k|k-1}^{(ij)}(n,m)\) as the \((n,m)\) element of the covariance matrix \(\mathbf{P}_{k|k-1}^{(ij)}\). Finally, we set \(\beta = -0.4\) where \(|\beta| \leq 1\) relates to the correlation between parent and daughter.

B. Results

The estimation results shown in Fig. 3 & 4 are from a simulation run with 500 particles for each target. Tracks are maintained by labelling each corresponding particle set. The state and error estimates are obtained by applying a similar clustering technique as demonstrated in [28] whereby the weights and means within the sum \(\sum_{\varphi \in \pi}\) in (27) are re-indexed as \(\omega_{\varphi,k}^{(n)}\) and \(\mathbf{m}_{\varphi,k}^{(n)}\) for \(n = 1, \ldots, N_{\varphi}\) where \(N_{\varphi} = J_{\xi}^{(i)} \times |\varphi| \times J_{\phi}^{(ij)}\).

Then, for each cluster subset \(\varphi\), we compute \(W_{\varphi,k} = \sum_{i=1}^{N_{\xi}} \sum_{n=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)}\), and if \(W_{\varphi,k} \geq T\), for some threshold \(T\), we calculate the position and shape
and the corresponding errors are calculated as
\[
\hat{\mathbf{m}}_{\varphi,k} = \sum_{i=1}^{N_{k-1}} \sum_{n=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{m}_{\varphi,k}^{(n)} , \quad \hat{\mathbf{s}}_{\varphi,k} = \sum_{i=1}^{N_{k-1}} \sum_{n=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{s}_{k}^{(n)} ,
\]
and the corresponding errors are calculated as
\[
\hat{\mathbf{m}}_{\varphi,k} = \sum_{i=1}^{N_{k-1}} \sum_{n=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{m}_{\varphi,k}^{(n)} , \quad \hat{\mathbf{s}}_{\varphi,k} = \sum_{i=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{s}_{k}^{(n)} ,
\]
and
\[
\hat{\mathbf{m}}_{\varphi,k} = \sum_{i=1}^{N_{k-1}} \sum_{n=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{m}_{\varphi,k}^{(n)} , \quad \hat{\mathbf{s}}_{\varphi,k} = \sum_{i=1}^{N_{\varphi}} \omega_{\varphi,k}^{(n)} \mathbf{s}_{k}^{(n)} .
\]
From the early simulation trials, it can be seen in Fig. 3 & 4 that the extracted estimates for both position and shape parameters are accurate, even when the uncertainty is high. The exception being when the targets approach the perimeter of the observation region towards the end of their existence, in which case the extents are only partly visible.

V. Conclusion

The paper presented a hierarchical point process approach to tracking an extended target which gives rise to a structured set of measurements per time-step in the form of a geometric shape. The implementation of the corresponding PHD filter for the compound state, consisting of the target’s shape parametric and kinematic state, was based on a particle representation and a Gaussian mixture formulation respectively. The novelty of the approach is that it’s non-shape specific, and is applicable for structured sets of target generated measurements which take the form of any geometric shape that can be parameterised in a two-dimensional observation space. Demonstrated on a multiple extended target example where the extents are elliptical in shape and variable in size, early simulation results are promising.

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