Source Localization by TDOA with Random Sensor Position Errors - Part I: Static Sensors

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Abstract—This paper presents a study on source localization using time difference of arrival (TDOA) measurements from static sensors in the presence of random errors in sensor positions. We develop a constrained weighted least squares (CWLS) source localization method which incorporates the relationship between the source position and an auxiliary variable as a constraint. The CWLS source localization is formulated as an indefinite quadratically constrained quadratic optimization problem, which is a nonconvex problem. By employing the hidden convexity of the original optimization problem, the global optimal source location estimate can be efficiently obtained. Simulations are used to corroborate the good performance of the proposed method.

Index Terms—Source localization, time difference of arrival, weighted least-squares, nonconvex problem, constrained weighted least-squares.

I. INTRODUCTION

The source localization problem has received significant attention in signal processing community literature due to its important applications in many areas such as target tracking, wireless communications and sensor networks [1], [2]. In this paper, we consider source localization by a network of static passive sensors, which arises naturally in both civilian and military applications whenever a signal-emitting target needs to be localized. We focuses on localization with the time difference of arrival (TDOA) measurements for a stationary source. The TDOA information depends directly on the source location relative to the sensor locations. Source localization with TDOA is not a trivial problem since the TDOA measurements are nonlinearly related to the source location.

There are many existing methods for the TDOA based localization. The first approach is the nonlinear least-squares (NLS) method based on the Taylor-series expansion for linearization [3], where a solution is obtained in an iterative manner. The second approach is a closed-form solution which rearranges the nonlinear TDOA equations into a set of linear equations by squaring them and introducing an auxiliary variable that depends on the source position. The closed-form solution is derived by the weighted least-squares (WLS) minimization. This reorganization idea was initially proposed in [4], and improved in [5]–[7] which make use of the relationship between the unknown source position and an auxiliary variable to achieve better localization. The closed-form solution is attractive because it does not require an initial guess close to the true source location and does not suffer from the local convergence problem.

However, all of the aforementioned methods are based on the assumption that the sensor positions are known exactly. In practice, the sensor locations may not be accurately known. The phenomena of sensor position uncertainty has already led researchers to tackle source localization problems by taking into account of the sensor position errors [8]–[10]. As illustrated in [8], even a small sensor position error can lead to significant degradation in the source localization accuracy. In [8]–[10], sensor position errors are modeled as independent Gaussian distributed noises, and taken into account in the weighting matrix of a two-step closed-form WLS source localization method. The first step is to estimate the unknown source position and the auxiliary variable with the weighted least-squares method by treating them as independent variables. The second step is to refine the solution obtained in the first step by using the relationship between the source position and the auxiliary variable.

In this paper, we consider incorporating the relationship between the source position and the auxiliary variable as a second order equality constraint in the weighted least-square optimization to improve the source localization accuracy in the presence of random sensor position errors. This results in a nonconvex indefinite quadratically constrained quadratic programming (QCQP). However, under a suitable simultaneous diagonalization assumption, this optimization problem can be equivalently transformed to a convex minimization problem due to the hidden convexity. We verify that the simultaneous diagonalization assumption is satisfied when the number of sensors is at least 5. This requirement is generally met in applications since at least four sensors are needed for source localization in a three-dimensional (3-D) space.

The rest of the paper is organized as follows. Section II formulates the localization problem and introduces the symbols and notations used. Section III presents the proposed centralized constrained weighted least-square source localization. Section IV is the evaluation of the Cramer-Rao lower bound (CRLB) as a performance benchmark and Section V contains the simulation results to support the theoretical development. Section VI is the conclusion.
For notation simplicity, we collect available position of sensor which are modeled as: measurements can be easily converted to range difference of reference sensor are available. Without loss of generality, let \( Q_s = [\Delta s', \Delta s', \cdots, \Delta s'] \) and the corresponding error vector \( \Delta s = [\Delta s', \Delta s', \cdots, \Delta s'] \), which is assumed to be zero-mean Gaussian with covariance matrix \( Q_s \).

Assuming line-of-sight signal propagation, the TDOA measurements of the received signals with respect to the signal at reference sensor are available. Without loss of generality, let the first sensor be the reference and the TDOA measurements model is given by

\[
t_{i1} = t_{i1}^0 + \Delta t_{i1},
\]

where \( t_{i1} \) is the estimated TDOA between sensor pair \( i \) and 1, \( t_{i1}^0 \) is the true TDOA and \( \Delta t = [\Delta t_{21}, \cdots, \Delta t_{n1}]' \) is zero-mean Gaussian noise with covariance \( Q_t \). The TDOA measurements can be easily converted to range difference of arrival (RDOA) measurements given the propagation speed \( c \), which are modeled as:

\[
r_{i1} = r_{i1}^0 + c \Delta t_{i1},
\]

where \( r_{i1} = ct_{i1}, r_{i1}^0 = ct_{i1}^0 \),

\[
r_{i1}^0 = r_{i1}^0 - r_{i1}, \quad \text{and}
\]

\[
r_{i} = ||u - s_i||, \quad r_{i1} = ||u - s_{i1}||.
\]

For notation simplicity, we collect \( r_{i1}, i = 2, 3, \cdots, n \) to form an \((n-1) \times 1\) RD measurement vector as

\[
r = [r_{21}, r_{31}, \cdots, r_{n1}]' = r^0 + \Delta r,
\]

where the RDOA error vector \( \Delta r = c \Delta t \) is zero-mean Gaussian noise with covariance \( c^2 Q_t \). In this paper, TDOA and RDOA will be used interchangeably.

The sensor position noise \( \Delta s \) is assumed to be independent of the TDOA noise \( \Delta t \). If the two kinds of noises are from different sources, this assumption is satisfied. We adopt this assumption here for ease of illustration. The analysis and the proposed solution can be obtained with slight modifications in the case of a correlated noise. Under such an source localization framework, the following questions are naturally raised:

- For a static sensor network, given the noisy TDOA measurements \( r \) together with the noisy sensor positions \( s \), how to estimate the source location \( u \) as accurately as possible?
- For a mobile sensor network, given the noisy TDOA measurements \( r^k \) together with the noisy sensor positions \( s^k \) in each sampling time step \( k = 1, 2, \cdots, \), how to fuse these information efficiently to estimate the source location \( u \)?

The objective of this paper is to answer the first question and the second one is to be investigated in a companion paper.

### III. The Constrained Weighted Least Square Source Localization

We shall develop a centralized constrained least-squares source localization method in the presence of Gaussian sensor position errors. Compared to the two stage WLS close-form localization method which exploits the relationship between \( u \) and \( r_1 \) implicitly via a relaxation procedure [5], [8], [9], the proposed method deals with the weighted least-squares estimation by explicitly incorporating the relationship between \( u \) and an auxiliary variable as a second order equality constraint. Our method is also different with the previous constrained least-squares method based on the technique of Lagrange multipliers [6], which does not take into account the uncertainty of the sensor locations as well as weighting matrix in the optimization, and needs to find the roots of a polynomial of degree six. We shall solve the resulted constrained WLS estimation efficiently by convex optimization.

#### A. Formulation of CWLS Source Localization

According to (4), the true RDOA \( r_{i1}^0 \) is related to the true ranges \( r_i^0 \) and \( r_1^0 \) as \( r_{i1}^0 + r_1^0 = r_{i1}^0 \). Squaring both sides and substituting \( r_{i1}^0 = (x-x_1^0)^2 + (y-y_1^0)^2 + (z-z_1^0)^2 \), the RDOA measurement equations can be simplified as:

\[
r_{i1}^2 - r_{i1}^2 - r_1^2 = -2(s_i^0 - s_1^0)' T u - 2r_{i1}r_1^0, \quad (7)
\]

where \( R_i^0 = \sqrt{x_i^2 + y_i^2 + z_i^2} \), and \( i = 2, \cdots, n \).

It can be seen that (7) is a set of linear equations with respect to \( u \) and \( r_{i1}^0 \). Actually, since \( r_{i1}^0 \) is the true distance between the unknown source and sensor 1, it depends on the unavailable \( s_i^0 \). Thus, we use the Taylor-series expansion to expand \( r_{i1}^0 \) around the noisy sensor position \( s_{i1} \) up to linear error term,

\[
r_{i1}^0 = ||u - s_{i1}|| = ||u - s_1 + \Delta s_1|| \\
\approx r_1 + s_{u,s_1}^T \Delta s_1
\]
where \( r_1 = \| \mathbf{u} - \mathbf{s}_1 \| \) and \( \mathbf{g}_{\mathbf{u}, \mathbf{s}_1} = \frac{\mathbf{u} - \mathbf{s}_1}{\| \mathbf{u} - \mathbf{s}_1 \|} \).

Let \( \mathbf{u}_1 = [x, y, z, r_1]' \) be the unknown vector. In terms of TDOA measurement noise and the sensor position errors by putting \( r_1^0 = r_{\mathbf{i}i} - c\Delta t_{\mathbf{i}i} \) and \( \mathbf{s}_1^0 = \mathbf{s}_i - \Delta \mathbf{s}_i \), the set of equations (7) can be rewritten as

\[
\epsilon = \mathbf{h} - \mathbf{G}\mathbf{u}_1
\]

where

\[
\mathbf{h} = \begin{pmatrix}
    r_{21}^0 - R_2^2 + R_1^2 \\
    \vdots \\
    r_{n1}^0 - R_n^2 + R_1^2
\end{pmatrix},
\]

\[
\mathbf{G} = -2 \begin{pmatrix}
    (\mathbf{s}_2 - \mathbf{s}_1)^T & r_{21}^0 \\
    \vdots & \vdots \\
    (\mathbf{s}_n - \mathbf{s}_1)^T & r_{n1}^0
\end{pmatrix},
\]

\[
R_i = \sqrt{x_i^2 + y_i^2 + z_i^2}.
\]

The equation error vector \( \epsilon \) can be written in terms of \( \Delta \mathbf{t} \) and \( \Delta \mathbf{s} \) as

\[
\epsilon = \mathbf{c}\mathbf{B}\Delta \mathbf{t} + \mathbf{c}^2 \Delta \mathbf{t} \otimes \Delta \mathbf{t} + \mathbf{D}\Delta \mathbf{s} + \Delta \mathbf{s} \otimes \Delta \mathbf{s}
\]

\[
\approx \mathbf{c}\mathbf{B}\Delta \mathbf{t} + \mathbf{D}\Delta \mathbf{s}
\]

where the second order error terms have been ignored and \( \otimes \) represents the element by element multiplication. The matrix \( \mathbf{B} \) is given by

\[
\mathbf{B} = 2 \begin{pmatrix}
    r_2^0 & 0 & \cdots & 0 \\
    0 & r_3^0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & r_n^0
\end{pmatrix},
\]

and \( \mathbf{D} \) is shown at the bottom of the next page in (12).

Since \( \epsilon_1 \) is approximated as a linear combination of Gaussian noises \( \Delta \mathbf{t} \) and \( \Delta \mathbf{s} \) with known covariance matrices \( \mathbf{Q}_t \) and \( \mathbf{Q}_s \), and \( \Delta \mathbf{t} \) is independent of \( \Delta \mathbf{s} \), the minimum variance unbiased (MVU) estimator of the linear model (9) is the maximum likelihood estimator, which is also equivalent to the weighted LS estimator. Meanwhile, there is additional side information available about the relationship between unknowns \( \mathbf{u} \) and \( r_1 \), which can be exploited to improve the location accuracy of the unknown source. Therefore, the localization problem with all the available information can be formulated as the following optimization problem:

\[
\min \frac{1}{2} \| \mathbf{h} - \mathbf{G}\mathbf{u}_1 \|^2 \mathbf{W}^{-1} \frac{1}{2} \| \mathbf{h} - \mathbf{G}\mathbf{u}_1 \|
\]

\[
s.t. \quad (\mathbf{u}_1 - \mathbf{s}_1)' \mathbf{C}(\mathbf{u}_1 - \mathbf{s}_1) = 0
\]

where \( \mathbf{C} = \text{diag}([1, 1, 1, -1]), \mathbf{s}_1 = [x_1, y_1, z_1, 0]' \) and \( \mathbf{W} \) is the weighting matrix defined as

\[
\mathbf{W} = \mathbf{E}[\epsilon_1\epsilon_1]'^{-1}.
\]

The strategy of incorporating the relationship between \( \mathbf{u} \) and \( r_1 \) as a second order equality constraint in the weighted LS estimator is called CWLS source localization.

B. Solution of the CWLS Source Localization

The CWLS source localization problem (13) is actually a quadratically constrained quadratic problem. Moreover, since \( \mathbf{C} = \text{diag}([1, 1, 1, -1]) \) is an indefinite matrix, the constraint is nonconvex. Fortunately, there is some hidden convexity in this nonconvex quadratically constrained quadratic program. Ben-Tal and Teboulle [11] discovered that under a suitable simultaneous diagonalization assumption the indefinitely quadratically constrained quadratic problem (QCQP) is equivalent to a convex minimization problem with simple linear constraints. Further, the explicit nonlinear transformation allows for recovering the optimal solution of the nonconvex original problem via its equivalent convex counterpart. In order to make the CWLS source localization problem (13) conformable with the model in [11], we reformulate problem (13) as follows:

\[
\min \mathbf{u}_2'\mathbf{G}_2\mathbf{u}_2 - 2\mathbf{h}_2'\mathbf{u}_2
\]

\[
s.t. \quad \mathbf{u}_2'\mathbf{C}\mathbf{u}_2 = 0
\]

where

\[
\mathbf{u}_2 = \mathbf{u}_1 - \mathbf{s}_1,
\]

\[
\mathbf{G}_2 = \mathbf{G}'\mathbf{W}\mathbf{G},
\]

\[
\mathbf{h}_2 = \mathbf{G}'\mathbf{W}(\mathbf{h} - \mathbf{G}\mathbf{s}_1).
\]

Lemma 1 (Ben-Tal and Teboulle [11]): Assume problem (15) is feasible and the simultaneous diagonalization holds for \( \mathbf{G}_2 \) and \( \mathbf{C} \), i.e., there exists a nonsingular matrix \( \mathbf{S} \) such that

\[
\mathbf{S}'\mathbf{G}_2\mathbf{S} = \mathbf{M} := \text{diag}(m_1, \cdots, m_4),
\]

\[
\mathbf{S}'\mathbf{C}\mathbf{S} = \mathbf{N} := \text{diag}(n_1, \cdots, n_4),
\]

the indefinite quadratic problem (15) is equivalent to the following convex program:

\[
\min \sum_{j=1}^4 m_j v_j - 2|a_j|\sqrt{v_j}
\]

\[
s.t. \quad \sum_{j=1}^4 n_j v_j = 0, \quad v \in \mathbb{R}_4^+,
\]

where \( \mathbf{S}'\mathbf{h}_2 = \mathbf{a} := [a_1, \cdots, a_4]' \).

Moreover, denote the optimal solution of problem (19) as \( \mathbf{v}^* = [v_1^* , \cdots , v_4^* ] \), for which a corresponding solution of (15) is given by

\[
\mathbf{u}_2^* = \mathbf{S}\mathbf{v}^*, \quad c_j = \text{sgn}(a_j)(v_j^*)^4, \quad j = 1, \cdots, 4,
\]

where \( \text{sgn}(a_j) = 1 \) if \( a_j > 0 \), \( \text{sgn}(a_j) = 0 \) if \( a_j = 0 \) and \( \text{sgn}(a_j) = -1 \) if \( a_j < 0 \).

Lemma 1 provides an efficient way to solve the CWLS source localization problem (15), but only when the assumption of simultaneous diagonalization is satisfied. The next lemma provides a sufficient condition for simultaneous diagonalization [12].

Lemma 2 (Horn and Johnson [12]): Let \( \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n} \) be two symmetric matrices and suppose that there exist \( \alpha, \beta \in \mathbb{R} \) such that

\[
\alpha \mathbf{A} + \beta \mathbf{B} \succ 0,
\]

then there exists a nonsingular matrix \( \mathbf{C} \in \mathbb{R}^{n \times n} \) such that both \( \mathbf{C}'\mathbf{A} \mathbf{C} \) and \( \mathbf{C}'\mathbf{B} \mathbf{C} \) are diagonal.
The sufficient condition in Lemma 2 is satisfied when either $A$ or $B$ is positive definite. Since $C$ in (15) is indefinite, we have to verify the symmetric matrix $G_2 \in \mathbb{R}^{4 \times 4}$ is positive definite in general cases. The weighting matrix $W$ is positive definite with rank $n - 1$, so $G_2$ is semi-positive definite with rank($G_2$) ≤ $n - 1$. The rank of $G$ depends on the available positions of the sensors $s$. Also, the relationship between the last column of $G$ and the first three columns is nonlinear, so generally the columns of $G$ can be linearly independent, and $G$ can have a full column rank of $n$. Therefore, the rank of $G_2$ can be 4 if $n \geq 5$, which is generally satisfied since at least four sensors are needed for source localization in three-dimensional (3-D) cases.

The following algorithm summarizes the procedure for simultaneous diagonalization of the matrices $G_2$ and $C$ in (15):

**Algorithm 1:** Simultaneous diagonalization

1) Perform the eigenvalue decomposition on $G_2$, which produces a vector $d$ of eigenvalues and a matrix $V$ whose columns are the corresponding eigenvectors.
2) Let $V_1 = V\text{diag}(e)$, where $e(i) = \sqrt{d(i)}$.
3) Do eigenvalue decomposition on $V_1^T C V_1$ and get the matrix $V_2$ with the corresponding eigenvectors.
4) Let $S = V_1 V_2$, then $S'G_2 S$ and $S'CS$ are diagonal.

We now summarize the CWLS source localization method: i) rearrange the RDOA equations as in (8), ii) formulate the CWLS source localization problem as in (15), iii) simultaneously diagonalize the matrices $G_2$ and $C$ following Algorithm 1, iv) solve the corresponding convex optimization problem and obtain a source location estimate from (20) and (16).

**Remark 1:** Note that in practice, the computation of the weighting matrix $W$ requires the true source location which is unavailable. Following a similar approach in [8] and [9], we could first set $W = (c^2Q_1)^{-1}$ and use the WLS method which ignores the constraint in (13) to obtain an initial estimate of $u$. It is then used to generate an improved weighting matrix $W$. We can repeat the above process to obtain satisfactory $W$ by one to three iterations. In addition, our extensive simulation results indicate that the location accuracy is relatively insensitive to the approximation of the weighting matrix in this manner and the performance degradation is insignificant.

**IV. CRAMER-RAO LOWER BOUND (CRLB)**

The CRLB establishes a lower bound on the error covariance matrix for any unbiased estimator. We can calculate the CRLB from the inverse of the Fisher information matrix (FIM) which is created from the probability density function (PDF) of the underlying problem. Since both the TDOA measurement noise $\Delta t$ and the sensor position noise $\Delta s$ are Gaussian distributed, and they are independent of each other, the logarithm of the probability density function of the available data vector $v = [r', s']$ on the unknowns $u$ and $s^0$ is:

$$
\ln p(v; u, s^0) = \ln p(r; u, s^0) + \ln p(s; u, s^0)
= k - \frac{1}{2\pi^2} (r - r^0)Q_1^{-1}(r - r^0) - \frac{1}{2} (s - s^0)^T Q_s^{-1} (s - s^0)
$$

(21)

where $k$ is a constant that does not depend on the unknowns. Applying partial derivatives with respect to the unknowns twice, negating the sign and then taking expectation, we have

$$
\text{FIM} = \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix}
$$

(22)

where

$$
X = -E \left( \frac{\partial^2 \ln p}{\partial u \partial u} \right) = \begin{pmatrix} g'_{u,s^0} - g'_{u,s} \\ \vdots \\ g'_{u,s^0} - g'_{u,s} \end{pmatrix},
$$

$$
Y = -E \left( \frac{\partial^2 \ln p}{\partial u \partial s^0} \right) = \begin{pmatrix} g'_{u,s^0} \\ \vdots \\ g'_{u,s^0} \end{pmatrix},
$$

$$
Z = -E \left( \frac{\partial^2 \ln p}{\partial s \partial s} \right) = \begin{pmatrix} (c^2Q_1)^{-1} \partial^2 \ln p/\partial s^2 \\ \vdots \\ (c^2Q_1)^{-1} \partial^2 \ln p/\partial s^2 \end{pmatrix} + Q_s^{-1}.
$$

(23)

The partial derivatives $\partial r^0 / \partial u$ and $\partial r^0 / \partial s^0$ are given as follows:

$$
\frac{\partial r^0}{\partial u} = \begin{pmatrix} g'_{u,s^0} - g'_{u,s} \\ \vdots \\ g'_{u,s^0} - g'_{u,s} \end{pmatrix},
$$

$$
\frac{\partial r^0}{\partial s^0} = \begin{pmatrix} g'_{u,s^0} \\ \vdots \\ g'_{u,s^0} \end{pmatrix},
$$

(24)

(25)

where $g_{u,s^0} = \frac{u - s^0}{\|u - s^0\|}$. Invoking the partitioned matrix inversion formula [13], the CRLB of the unknown $u$ is

$$
\text{CRLB}(u) = (X - YZ^{-1}Y')^{-1} = X^{-1} + X^{-1}Y(Z - Y'X^{-1}Y)^{-1}Y'X^{-1}
$$

(26)

Notice that $X^{-1}$ is the CRLB of $u$ when there is no sensor position noise. Hence, the second term of (26) represents the increase of CRLB in the presence of $\Delta s$. The trace of

$$
D = 2 \times \begin{pmatrix} -r_{21}g'_{u,s} - (u - s_1)' (u - s_2)' & 0' & \cdots & 0' \\
-r_{31}g'_{u,s} - (u - s_1)' & 0' & \cdots & 0'T \\
\vdots & \vdots & \ddots & \vdots \\
-r_{n1}g'_{u,s} - (u - s_1)' & 0' & \cdots & (u - s_n)' \end{pmatrix}
$$

(12)
is the minimum possible source location MSE that any unbiased estimator can achieve. We shall use the CRLB as a performance benchmark of an estimator in the following simulations.

V. NUMERICAL SIMULATIONS

This section contains simulation results of the proposed CWLS source localization method and their comparison with the CRLB and the closed-form WLS method [9]. The simulation scenario contains \( n = 8 \) sensors, and their nominal positions are given in Table I. In generating the simulation results, TDOA measurements are obtained by adding Gaussian noise with covariance matrix \( \sigma^2Q_l \) to the true values, where \( Q_l = \sigma^2T \) and \( T \) is the \((n-1) \times (n-1)\) matrix with 1 in the diagonal elements and 0.5 otherwise. The sensor position noises at different locations and for different receivers are assumed to be identically independent Gaussian noises with variance \( \sigma^2_s \), i.e. \( Q_s = \sigma^2_sI \), where \( I \) is the \(3n \times 3n \) identity matrix. The TDOA noise and sensor position noise are independent. We set \( \sigma_2 = p \times 0.1 \) micro-second, \( \sigma_s = p \times 10/\sqrt{3} \) m, and \( p \) varies from 0.1 to 5.

<table>
<thead>
<tr>
<th>Sensor no.</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( z_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5000</td>
<td>5000</td>
<td>6000</td>
</tr>
<tr>
<td>2</td>
<td>5000</td>
<td>-5000</td>
<td>6000</td>
</tr>
<tr>
<td>3</td>
<td>-5000</td>
<td>-5000</td>
<td>6000</td>
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<tr>
<td>4</td>
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<td>5</td>
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</tr>
<tr>
<td>8</td>
<td>-5000</td>
<td>5000</td>
<td>1000</td>
</tr>
</tbody>
</table>

We consider a near-field unknown source located at \([20000, 50000, 0]^T\) m as well as a far-field source located at \([20000, 50000, 0]^T\) m. The implementation of the proposed CWLS sensor localization algorithm follows the steps as described in Section III. In order to get an accurate weight matrix \( W \) in the second stage, we apply the iterations in Remark 1 three times. The optimization problem of the fourth stage is solved by the CVX toolbox in MATLAB. The results for the closed-form WLS method [9] are also generated for comparison. The localization accuracy is evaluated by the average range error (ARE) and the standard deviation (SD) of localization error which are defined as

\[
\text{ARE}(u) = \frac{1}{L} \sum_{i=1}^{L} ||\hat{u}_i - u||_2 / L, \quad \text{SD}(u) = \sqrt{\frac{1}{L} \sum_{i=1}^{L} ||\hat{u}_i - u||_2^2 / L},
\]

where \( \hat{u}_i \) denotes the unknown source position estimate at ensemble \( l \) and \( L = 1000 \) is the number of ensemble runs.

Fig. 1 shows the localization accuracy of the proposed CWLS solution [denoted by cross symbols] and the closed-form WLS approach [denoted by diamond symbols] for the near-field source at \([20000, 10000, 0]^T\) m. It is evident from the figure that both the two methods are able to reach the CRLB accuracy for the near-field source when \( p \leq 3 \). The ARE and SD of our method when \( p = 0.1 \) is slightly larger than that of the closed-form WLS method, which is mainly because of the numerical problems in solving the convex optimization when the noise level is too low. When \( 3 < p \leq 4 \), the performance of the proposed method and the WLS method is comparable. However, as the TDOA noises and sensor noises increase such that \( p > 4 \), the proposed CWLS method achieves a significant performance improvement with respect to the closed-form WLS method.

Fig. 2 is the result for the far-field source located at \([20000, 50000, 0]^T\) m. The localization accuracy is generally worse for a far-field source than a near-field source. The performance of the proposed CWLS method is comparable with the closed-form WLS method. As expected from the theory, the theoretical MSE curves overlap with the CRLB before thresholding effect occurs. Here, the thresholding effect means the sudden deviation of the localization accuracy from the CRLB as the variance of the measurement noise increases. This is a consequence of the nonlinear nature of the source localization problem.

VI. CONCLUSION

This paper reinvestigated the source localization problem using TDOA measurements with random sensor position errors in static sensor network. The CWLS localization method was approached by rearranging the nonlinear TDOA equations into a set of linear equations while introducing an auxiliary variable. We directly incorporated the relationship between the unknown source and the auxiliary variable as a constraint to the WLS strategy. The resulted nonconvex QCQP problem is equivalently transformed to a convex problem by the hidden convexity, so the global optimal solution can be effectively achieved. Judging from the numerical simulation results, the CWLS localization can reach the CRLB accuracy when the TDOA noises level and sensor position noises level are small. In the far-field cases, the proposed CWLS method is comparable with the closed-form WLS method, but in the near-field cases the CWLS method can significantly improve the localization accuracy under high noises level.

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Fig. 1. Comparison of the localization accuracy of the proposed CWLS method and the closed-form WLS method for a near-field unknown source. The left is the ARE of location when $\sigma_t = 0.1\mu$s and $\sigma_s = 10\mu$/√3 m, and the right is the corresponding root of CRLB and SD.

Fig. 2. Comparison of the localization accuracy of the proposed CWLS method and the closed-form WLS method for a far-field unknown source. The left is the ARE of location when $\sigma_t = 0.1\mu$s and $\sigma_s = 10\mu$/√3 m, and the right is the corresponding root of CRLB and SD.