Advances on Tracking of Extended Objects and Group Targets using Random Matrices

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Abstract – The task of tracking extended objects or (partly) unresolvable group targets raises new challenges for both data association and track maintenance. Due to limited sensor resolution capabilities, group targets (i.e., a number of closely spaced targets moving in a coordinated fashion) may show a similar detection pattern as extended objects, namely a varying number of detections whose spread is determined by both the statistical sensor errors as well as the physical extension of the group or extended object. Different tracking approaches treating these situations have been proposed where physical extension is represented by a random symmetric positive definite matrix. This paper discusses some results that should give deeper insight into behavior and performance analysis of these approaches. Further improvements are presented.

Keywords: Target tracking, extended targets, group targets, formations, sensor resolution, random matrices, matrix-variate analysis.

1 Introduction

In many tracking applications, the objects to be tracked are considered as point sources, i.e., their extension is assumed to be neglectable in comparison with sensor resolution and error. With ever increasing sensor resolution capabilities however, this assumption is no longer valid, e.g., in short-range applications or for maritime surveillance where different scattering centers of the objects under consideration may give rise to several distinct detections varying, from scan to scan, in both number as well as relative origin location. From the associated data—assuming that the related association problem has been solved—one cannot only estimate the kinematic state of the object but also its extension (honoring the spread of the data in comparison with the expected statistical sensor error). But, more than these quantities cannot safely be estimated as well in the (opposite) case where limited sensor resolution causes a fluctuating number of detections for a group of closely spaced targets and thus prevents a successful tracking of (all of) the individual targets.

Several suggestions for dealing with this problem can be found in literature. For an early work, see [1], for an overview of existing work up to 2004, refer to [2]. In contrast to the probability hypothesis density (PHD) filter [3–7] that inherently does not differ between what part of the (hypothesis) density spread is due to estimation uncertainty and what amount is due to an actual physical extension of the extended object or group, the Bayesian approach suggested in [8], [9] does exactly this by trying to estimate both a kinematic state (a random vector) on the one hand and physical extension (represented by a random matrix) on the other. It does not, however, completely solve the association problem. Association is also beyond the scope of this paper and we rather concentrate on track maintenance as well. In order to circumvent some of the problems one may face when applying the Bayesian group tracking approach under circumstances where the underlying assumptions of [8], [9] do not hold, a new approach to tracking of extended objects and group targets using random matrices has been proposed in [10]. In the following, some of the very ad-hoc proposal made there will be replaced by results obtained from a more thorough analysis of the underlying statistical models.

The paper is organized as follows: We start with summarizing the two approaches of [8], [9], and [10]. Subsequently, we discuss some properties of the distribution used for representing object or group extension. Moment matching is analyzed in context with an interacting multiple model (IMM). After showing how to visualize the extent estimate and a corresponding confidence region, a simulation result is given.

2 Bayesian extended object tracking

As has been stated in the introduction, the Bayesian approach to tracking extended objects and group targets in [8], [9] adds to the kinematic state of the centroid described by the random vector $x_k$ the physical extension represented by a symmetric positive definite (SPD) random matrix $X_k$ thus considering some ellipsoidal shape. It is assumed that in each scan $k$ there are $n_k$ independent position measurements

$$y_k^j = H x_k + w_k^j$$  \hspace{1cm} (1)

where the random vector $x_k$ denotes the state to be estimated (for us, position and velocity in two or three spatial dimen-
sions, i.e., $x_T = \{r_T, \dot{r}_T\}$, but one may add acceleration $\ddot{r}_T$ here, of course) and $y_k$ the actual measurement (in the following, position only, i.e., $H = [I_d, 0_d]$ with $d = 2, 3$).

In this paper, we will use the abbreviations $Y_k := \{y_j^k\}_{j=1}^{n_k}$ and $\mathcal{Y}_k := \{n_{X}, Y_{\mathcal{X}_k}\}_{k=0}^{\infty}$ to denote the set of the $n_k$ measurements in a particular scan and for the sequence of what is measured scan by scan, respectively.

Now, one decisive assumption in [8], [9] is that the statistical error of each individual measurement $y_j^k$ is small and thus neglectable in comparison with the object extension that hence dominates the spread of the measurements. In detail, the measurement noise $w^k_j$ is assumed to be a zero mean normalized random vector with variance $X_k$. With this, the likelihood to measure the set $Y_k$ given both kinematic state and extension as well as the number of measurements, reads

$$p(Y_k|n_k, x_k, X_k) = \prod_{j=1}^{n_k} N(y_j^k; Hx_k, X_k)$$

(2)

where

$$N(x; \mu, \Sigma) = \frac{\exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)}{\sqrt{2\pi} \sqrt{|\Sigma|}} \exp \left( -\frac{1}{2} \text{etr} \left( -\frac{1}{2} \Sigma X \Sigma^{-1} \right) \right)$$

(3)

denotes the normal density with mean $\mu$ and variance $\Sigma$. Introducing the mean measurement and the measurement spread

$$\overline{y}_k = \frac{1}{n_k} \sum_{j=1}^{n_k} y_j^k, \quad \overline{Y}_k = \sum_{j=1}^{n_k} (y_j^k - \overline{y}_k)(y_j^k - \overline{y}_k)^T$$

(4)

it is easily shown that eq. (2) can be written as

$$p(Y_k|n_k, x_k, X_k) \propto N(\overline{y}_k; H\overline{x}_k, \frac{X_k}{n_k}) \times W(Y_k; n_k - 1, X_k)$$

(5)

where

$$W(X; m, C) = \frac{|X|^\frac{m - d}{2}}{2^m \pi^\frac{m}{2} \Gamma_d (\frac{m}{2}) |C|^{-\frac{m}{2}}} \exp \left( -\frac{1}{2} \text{etr} \left( -\frac{1}{2} C X^{-1} \right) \right)$$

(6)

with $m \geq d$ denotes the Wishart density [11] of a $d$-dimensional SPD random matrix $X$ with expected SPD matrix $mC, \text{etr}()$ is an abbreviation for $\exp (\text{tr}(\cdot))$, and $\Gamma_d$ is the multivariate gamma function.

As has been shown in [8], [9], one may write a suitable conjugate prior $p(x_k, X_k|Y_{k-1})$ for (2) as the product of two densities

$$p(X_k|Y_{k-1}) = \mathcal{W} \left( X_k; \nu_k|k-1, \overline{X}_k|k-1 \right)$$

(7)

and

$$p(x_k|X_k, Y_{k-1}) = N(x_k; x_k|k-1, \overline{P}_k|k-1 \otimes X_k)$$

(8)

where $p(X_k|Y_{k-1})$ is the matrix-valued inverse Wishart density [11] with parameterization

$$\mathcal{W}(X; m, C) = \frac{|C|^\frac{m}{2}}{2^m \pi^\frac{m}{2} \Gamma_d (\frac{m}{2}) |X|^{\frac{m + d}{2}}} \exp \left( -\frac{1}{2} C X^{-1} \right)$$

(9)

and expected SPD matrix $C/(m - d - 1)$ for $m - d - 1 > 0$ while $\otimes$ denotes the Kronecker product [12] that, e.g., yields

$$\begin{bmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \end{bmatrix} \otimes X_k = \begin{bmatrix} p_{11}X_k & p_{12}X_k \\ p_{21}X_k & p_{22}X_k \end{bmatrix}$$

(10)

This approach for the prior indeed results in a posterior of analogue form.

The kinematics update equations stemming from this prior strongly resemble standard Kalman updates. Writing $H = [I_d, 0_d] = \tilde{H} \otimes I_d$ with $\tilde{H} = [1, 0]$ one finds the updated parameters of the vector-variate density

$$x_k|k = x_k|k-1 + (K_k|k-1 \otimes I_d)(Y_k - Hx_k|k-1)$$

(12)

$$P_k|k = P_k|k-1 - K_k|k-1 S_k|k-1 K_k|k-1^T$$

(13)

$$S_k|k-1 = \tilde{H}P_k|k-1 \tilde{H}^T + \frac{n_k}{m}$$

(14)

$$K_k|k = \tilde{P}_k|k-1 \tilde{H}^T S_k|k-1^{-1}$$

(15)

and hence a scalar innovation variance $S_k|k-1$ while the update equations for the matrix-variate density can be shown to be

$$\nu_k|k = \nu_k|k-1 + n_k$$

(16)

$$\overline{X}_k|k = \overline{X}_k|k-1 + \tilde{S}_k|k-1 N_k|k-1 + \bar{Y}_k$$

(17)

$$N_k|k-1 = \left( Y_k - Hx_k|k-1 \right) \left( Y_k - Hx_k|k-1 \right)^T$$

(18)

The corresponding kinematics prediction equations given in [8], [9] read

$$x_k|k-1 = (\tilde{F} \otimes I_d)x_k|k-1$$

(19)

$$\tilde{P}_k|k-1 = \tilde{F}P_k|k-1 \tilde{F}^T + \tilde{Q}$$

(20)

where $\tilde{F}$ and $\tilde{Q}$ are state transition matrix and process noise variance, respectively, of a corresponding movement in one spatial dimension. Assuming that the extension does not tend to change over time and postulating $X_k|k-1 = X_k|k-1|k-1$ with

$$X_k|k-1 := E [X_k|Y_{k-1}] = \frac{\overline{X}_k|k-1}{\nu_k|k-1 - d - 1}$$

(21)

plus an increased uncertainty about the extension estimate, the heuristic prediction equations

$$\overline{X}_k|k-1 = \frac{\nu_k|k-1 - d - 1}{\nu_k|k-1 - d - 1} \overline{X}_k|k-1|k-1$$

(22)

$$\nu_k|k-1 = \exp \left( -\frac{T}{\tau} \right) \nu_k|k-1|k-1$$

(23)

have been proposed in [8], [9]. Here, $T$ is the prediction time interval while $\tau$ denotes some time constant related to the agility with which the object may change its extension over time (where a higher $\tau$ means a less agile object). In order to wrap up this section, it should be noted that, while (7) already shows the marginal density $p(X_k|Y_{k-1})$, the marginal $p(x_k|Y_k) = \int p(x_k, X_k|Y_k) dX_k$ turns out to have a Student-t density [11] from which, in particular, the mean squared estimation error can be deduced.
3 A different approach

In [10], several implications of the neglection of any (unavoidable) statistical sensor error in eq. (1) have been brought to attention. With this assumption, there is an almost fixed (up to a scalar constant) coupling between the estimated extension and the mean squared position estimation error. Although both the prediction equations (19/20) and the update equations (12/13) of the kinematic state look similar to that of the Kalman filter, the parameter \( \hat{P}_{k|t} \) therein behaves as the solution of the tracking problem in one spatial dimension. Consequently, one gets the same effective Kalman gain in each spatial dimension, independently of the actual sensor performance. And, eq. (20) implies that the effective process noise \( Q \otimes X_k \) driving the centroid kinematics depends on the size of the object in a very specific way which means that larger objects become more reactive. If sensor errors become significant, the algorithm effectively estimates extension plus sensor error which consequently leads to a more than proportionally increased centroid estimation error.

In view of these observations, a new approach has been sought that allows reliable tracking of extended objects and group targets in cases where sensor errors cannot be ignored when compared with object or group extension. The proposed alternative approach in [10] honors the fact that both sensor error and extension contribute to the measurement spread. Simulation results have shown that this approach can compensate significant sensor errors to a large extent and, thus, although compensation is not complete, may be able to, e.g., detect orientation changes of formations in cases where the original approach of [8], [9] might fail to do so.

As an approximation to the true behavior, the alternative approach in [10] uses eq. (1) with measurement errors being normally distributed with variance \( X_k + R \) which leads to

\[
p(X_k|x_k, X_k) = \prod_{j=1}^{n_k} Hx_k, X_k + R)
\]

It appears that, for this likelihood, no conjugate prior can be found that is both independent of \( R \) and analytically traceable. Thus, some careful approximations are necessary.

It has been proposed to use, for the kinematics of the centroid, standard Kalman updates according to

\[
x_{k|k} = x_{k|k-1} + K_{k|k-1} \left[ y_k - Hx_{k|k-1} \right]^T
\]

\[
P_{k|k} = P_{k|k-1} - K_{k|k-1} S_{k|k-1} K_{k|k-1}^T
\]

\[
S_{k|k-1} = HP_{k|k-1}H^T + \frac{Y_{k|k-1}}{n_k}
\]

\[
K_{k|k-1} = P_{k|k-1}H^T S_{k|k-1}^{-1}
\]

\[
Y_{k|k-1} = X_{k|k-1} + R
\]

denotes the predicted variance of a single measurement. Herein, the uncertainty coming with the extension estimate is merely ignored while knowing very well that the computed matrix \( S_{k|k-1} \) thus in turn is only an approximation to the true innovation covariance. This proposal may be interpreted as approximating the posterior of the kinematic state conditioned on the extension by the respective (in fact unknown) marginalized density that in turn is assumed to be close to a normal density again, i.e.,

\[
p(x_k|y_k) \approx p(x_k|y_k) \approx \mathcal{N} \left( x_k; x_{k|k}, P_{k|k} \right)
\]

As before, it is assumed that the marginalized prior density of the extension is an inverse Wishart density as in eq. (7) and that the corresponding (also in fact unknown) posterior is again of the same form. As an update of the estimate \( X_{k|k} \) (rather than the parameter \( X_{k|k} \)), a weighted sum of the predicted extension \( X_{k|k-1} \), the term \( N_{k|k-1} \) of eq. (18) and the measurement spread \( Y_k \) of eq. (4) is used again. But, weighting of two of these terms here is performed by matrix-valued factors. Computing some square roots (e.g., via Cholesky factorization) of the matrices \( X_{k|k-1}, S_{k|k-1}, Y_{k} \) and \( Y_{k} \) that obey

\[
X_{k|k-1} = X_{k|k-1}^{1/2} (X_{k|k-1}^{1/2})^T
\]

etc., one defines

\[
\hat{N}_{k|k-1} = X_{k|k-1}^{1/2} S_{k|k-1}^{-1/2} N_{k|k-1} (S_{k|k-1}^{-1/2} X_{k|k-1}^{1/2})^T
\]

\[
\hat{Y}_{k|k-1} = X_{k|k-1}^{1/2} Y_{k|k-1}^{-1/2} Y_{k|k-1}^{-1/2} (X_{k|k-1}^{1/2})^T
\]

and with this performs the extension updates according to

\[
x_{k|k} = \frac{1}{\alpha_{k|k}} \left( \alpha_{k|k-1} X_{k|k-1} + \hat{N}_{k|k-1} + \hat{Y}_{k|k-1} \right)
\]

\[
\alpha_{k|k} = \alpha_{k|k-1} + n_k
\]

Note the similarity between the choice (34) and eq. (17) as well as between eqs. (35) and (16).

With the assumed (approximate) independence between the estimates for centroid kinematics and extension expressed in eq. (30) and further assuming independent kinematic models for both of them, the standard Kalman filter prediction equations

\[
x_{k|k-1} = Fx_{k-1|k-1}
\]

\[
P_{k|k-1} = FP_{k-1|k-1}F^T + Q
\]

are applicable while, owing to the same heuristics as in the previous section, it has been suggested to use

\[
X_{k|k-1} = X_{k|k-1}
\]

Now, we will see later on that the variance of the extension estimate is approximately proportional to \( 1/(\alpha_{k|k} - 2) \) for both very large \( \alpha_{k|k} \) as well as values of \( \alpha_{k|k} \) close to 2 where \( \alpha_{k|k} > 2 \) is required to hold. In view of this and modifying our proposal in [10], we suggest to assume either an exponential increase of variance over time according to

\[
\alpha_{k|k-1} = 2 + \exp \left( -T/\tau \right) \left( \alpha_{k|k-1} - 2 \right)
\]

or a linear increase stemming from

\[
\alpha_{k|k-1} = 2 + \frac{\alpha_{k|k-1} - 2}{1 + (T/\tau)(\alpha_{k|k-1} - 2)}
\]

which probably is more feasible in practical applications.
4 Estimation error and moment matching

The two approaches recalled in the previous sections share one common feature, namely the fact that the marginal for the extension is an inverse Wishart density. In the following, we will focus on this matrix-valued density and discuss some related aspects. In order to simplify notation, we will from now on omit the explicit conditioning on the measurements when computing expected values (writing, e.g., $E[X_k]$ instead of $E[X_k|Y_k]$). With the parametric representation (9), we thus shortly write

$$E[X_k] = X_{k|k} = \mathbf{X}_{k|k} \alpha_{k|k}$$ with $\alpha_{k|k} = \nu_k - d - 1$ (41)

Now, let $\mathbf{X}_k$ denote any estimate for the positive definite symmetric matrix $\mathbf{X}_k$. The mean square error (MSE) $\hat{\epsilon}$ of this estimate is computed by summing up the mean square errors obtained for each element of $\mathbf{X}_k$ with respect to $\mathbf{X}_k$. As both $\mathbf{X}_k$ and $\mathbf{X}_k$ are SPD matrices, this error can be written as

$$\hat{\epsilon}_{k|k} = \text{tr} E[(\mathbf{X}_k - \mathbf{X}_k)^2]$$

$$= \text{tr} E[(\mathbf{X}_k - \mathbf{X}_{k|k})^2] + \text{tr} ((\mathbf{X}_k - \mathbf{X}_{k|k})^2)$$

$$- \text{tr} E[(\mathbf{X}_k - \mathbf{X}_{k|k})|(|\mathbf{X}_k - \mathbf{X}_{k|k}|)]$$

$$- \text{tr} ((\mathbf{X}_k - \mathbf{X}_{k|k}) E[|\mathbf{X}_k - \mathbf{X}_{k|k}|]) (42)$$

With $E[\mathbf{X}_k - \mathbf{X}_{k|k}] = 0$, this expression becomes minimal for $\mathbf{X}_k = \mathbf{X}_{k|k}$. In other words, the conditional mean is the MMSE estimator (just as it is in the vector-variate case) and its squared estimation error $\hat{\epsilon}_{k|k}$ can be computed from the variance

$$\text{Var}[\mathbf{X}_k] := E[(\mathbf{X}_k - \mathbf{X}_{k|k})^2] = E[\mathbf{X}_k^2] - \mathbf{X}_{k|k}^2$$

(44)

according to

$$\hat{\epsilon}_{k|k} = \text{tr} \text{Var}[\mathbf{X}_k]$$

(45)

For a random SPD matrix following an inverse Wishart distribution, the variance is given by [11]

$$V_{k|k} := \text{Var}[\mathbf{X}_k] = \frac{\alpha_{k|k} \text{tr} (\mathbf{X}_{k|k}) \mathbf{X}_{k|k} + (\alpha_{k|k} + 2) \mathbf{X}_{k|k}^2}{(\alpha_{k|k} + 1)(\alpha_{k|k} - 2)}$$

(46)

which yields the MMSE

$$\hat{\epsilon}_{k|k} = \frac{\alpha_{k|k} \text{tr} (\mathbf{X}_{k|k})^2 + (\alpha_{k|k} + 2) \text{tr} (\mathbf{X}_{k|k}^2)}{(\alpha_{k|k} + 1)(\alpha_{k|k} - 2)}$$

(47)

for $\alpha_{k|k} > 2$.

With these results, it becomes apparent that a moment matching, i.e., the process of determining the parameters of the inverse Wishart distribution from given first and second order moments, leaves only the single free parameter $\alpha_{k|k}$ once the expected matrix $\mathbf{X}_{k|k}$ has been fixed. Hence, one cannot expect in general to be able to fulfill eq. (46) completely. What we can do is to ensure eq. (47) being valid. Starting from there, we first obtain a quadratic equation

$$\hat{\epsilon}_{k|k}^2 + p_{k|k}\alpha_{k|k} - q_{k|k} = 0$$

(48)

and next the solution

$$\alpha_{k|k} = \frac{p_{k|k} + \sqrt{p_{k|k}^2 + 4q_{k|k}q_{k|k}}}{2q_{k|k}}$$

(49)

with

$$p_{k|k} = \epsilon_{k|k} + \text{tr}(\mathbf{X}_{k|k}^2) + \epsilon_{k|k} > \epsilon_{k|k}$$

(50)

$$q_{k|k} = 2 \left( \epsilon_{k|k} + \text{tr}(\mathbf{X}_{k|k}^2) \right) > 2 \epsilon_{k|k}$$

(51)

Note that in (49) we cannot use the negative of the square root instead as this would yield some negative $\alpha_{k|k}$. On the other hand, the selected sign obviously ensures $\alpha_{k|k} > 2$ and thus a valid solution.

5 IMM

In this section, we are going to refine our very ad-hoc proposal of [10] for integrating the alternate algorithm for tracking extended objects or group targets with random matrices into the well-known interacting multiple model (IMM) approach. Details about the derivation and the different processing steps of an IMM for point targets can be found at various places in literature, e.g., in [13]. Denoting by $\pi_k^i_{k-1|k-1}$ the probability of model $i$ (out of $R$) having been true a time $k - 1$, the processing cycle starts with the computation of the predicted model probabilities $\pi_k^i_{k-1|k-1} = \sum_{r=1}^{R} \pi_{k-1|k-1}^r \pi_{k-1|k-1}^r$ where $\pi_{k-1|k-1}^r$ is the probability of transitioning from model $i$ to model $j$. Next, the mixing probabilities $\pi_{k-1|k-1}^r = \pi_{k-1|k-1}^r \pi_{k-1|k-1}^r / \pi_{k-1|k-1}^r$ are computed. With these, we perform the standard interaction step

$$\mathbf{X}_{k|k-1}^0 = \sum_{i=1}^{R} \pi_{k-1|k-1}^i \mathbf{X}_{k|k-1}^i$$

(52)

$$\mathbf{P}_{k|k-1}^0 = \sum_{i=1}^{R} \pi_{k-1|k-1}^i \left\{ \mathbf{P}_{k-1|k-1}^i \right\}$$

(53)

for the centroid kinematics estimate. This step can be motivated by replacing the true resulting (in our case marginal) Gaussian mixture densities by normal densities with the same first and second order moments. As has been mentioned, a corresponding matching of the second order moments cannot be fully done for the inverse Wishart densities that describe the extension. Yet, in view of the results above, a complete first order moment matching in combination with the scalar second order moment matching yields

$$\mathbf{X}_{k-1|k-1}^{0j} = \sum_{i=1}^{R} \pi_{k-1|k-1}^i \mathbf{X}_{k-1|k-1}^i$$

(54)

$$\epsilon_{k|k}^{0j} = \sum_{i=1}^{R} \pi_{k-1|k-1}^i \left\{ \epsilon_{k-1|k-1}^j \right\} + \text{tr} \left( \left( \mathbf{X}_{k-1|k-1}^i - \mathbf{X}_{k-1|k-1}^{0j} \right)^2 \right)$$

(55)
Starting from these initial values, each model is individually updated with the measurements. In order to proceed further, the model-dependent measurement likelihoods \( \Lambda_{k|k-1}^j = p(Y_{k|j}, \mathbf{X}_{k|k-1}) \) have to be computed next. Again, these are not known exactly, hence is has been proposed in [10] to use the (approximate) innovation likelihood of the mean measurement with respect to the centroid. Yet, this does not take into account how well the measurement spread matches the predicted extension. At this point, we propose a heuristic way how to do that. We introduce
\[
\Delta_{k|k-1}^j := \mathbf{Y}_k - (n_k - 1)(\mathbf{X}_{k|k-1}^j + \mathbf{R})
\]
(56)
with
\[
\Delta_{k|k-1}^j := \Delta_{k|k-1}^j + (n_k - 1)(\mathbf{X}_k - \mathbf{X}_{k|k-1})
\]
(57)
The variance of \( \Delta_{k|k}^j \) for given \( \mathbf{X}_k \) can be obtained from the Wishart density \( \mathcal{W}(\mathbf{Y}_k; n_k - 1, \mathbf{X}_k + \mathbf{R}) \) using a result in [11] that, replacing \( \mathbf{X}_k \) with the model-specific prediction \( \mathbf{X}_{k|k-1}^j \) afterwards, approximately is
\[
\text{Var}[\Delta_{k|k-1}^j] \approx (n_k - 1)(\text{tr}(\mathbf{Y}_{k|k-1}^j))^{-1} + (\mathbf{Y}_{k|k-1}^j)^2)
\]
(59)
where \( \mathbf{Y}_{k|k-1}^j \) has been introduced in eq. (29). In view of
\[
\text{Var}[\Delta_{k|k-1}^j] = \text{Var}[\Delta_{k|k}^j] + (n_k - 1)^2 \mathbf{V}_{k|k-1}^j
\]
(60)
with \( \mathbf{V}_{k|k-1}^j \) given as in eq. (46), we propose to use
\[
\Lambda_{k|k-1}^j \propto \mathcal{N}(\mathbf{Y}_k; \mathbf{X}_k, \mathbf{S}_{k|k-1}^j)
\]
(61)
for \( n_k \geq 2 \) and solely \( \Lambda_{k|k-1}^j \approx \mathcal{N}((\mathbf{Y}_k; \mathbf{Hx}_k, \mathbf{S}_{k|k-1}^j) \) in case of a single measurement.

Now, one obtains the updated model likelihoods according to
\[
\pi_{k|k}^j = \Lambda_{k|k-1}^j \pi_{k|k-1}^j / \sum_{j=1}^{R} \Lambda_{k|k-1}^j \pi_{k|k-1}^j
\]
(62)
in final computing, for output purposes only, the mixed estimates from the updated model estimates in analogy to eqs. (52) to (55) with \( \pi_{k|k-1}^j \) replaced by \( \pi_{k|k}^j \).

6 Confidence regions

When using estimators, one usually is not only interested in the estimate itself but also in the corresponding MSE. For the kinematics estimate \( \mathbf{x}_{k|k} \), this MSE is given by \( \text{tr Var}[\mathbf{x}_k] \) and, as we have seen above, for the extension estimate likewise by \( \text{tr Var}[\mathbf{X}_k] \). More detailed information than the respective numbers alone can be envisioned by displaying certain confidence regions. For the kinematics estimate, this is rather standard. With \( \text{Var}[\mathbf{x}_k] = \mathbf{P}_{k|k} \) and a prescribed confidence value \( c \), one computes a height level \( h = h(c) \) such that
\[
P\{ (\mathbf{x}_{k|k} - \mathbf{x}_k)^T \mathbf{P}_{k|k}^{-1} (\mathbf{x}_{k|k} - \mathbf{x}_k) \leq h \} = c
\]
(63)
where \( h(c) \) can be determined numerically under normality assumptions because \( (\mathbf{x}_{k|k} - \mathbf{x}_k)^T \mathbf{P}_{k|k}^{-1} (\mathbf{x}_{k|k} - \mathbf{x}_k) \) then possesses a \( \chi^2 \) distribution. The ellipse given by all values \( \mathbf{c}_{k|k} \) obeying
\[
\mathbf{a}_{k|k}^T \mathbf{P}_{k|k}^{-1} \mathbf{a}_{k|k} = h
\]
(64)
thus is the border of a region in that, for the reported estimate \( \mathbf{x}_{k|k} \), the true value \( \mathbf{x}_k \) will lie with probability \( c \) as shown in Fig. 1.

Now, we will discuss how one can envision a confidence region for the extension estimate, too. We start with the observation that, although \( \text{Var}[\mathbf{X}_k] \) is again a positive definite symmetric matrix, the meaning of any ellipse defined by this matrix (formally replacing \( \mathbf{P}_{k|k} = \text{Var}[\mathbf{x}_k] \) by \( \text{Var}[\mathbf{X}_k] \) in eq. (63)) remains unclear, especially in view of the fact that the resulting \( \mathbf{c}_{k|k} \) would have a physical unit of an area \( (m^2) \) or the like. Instead of doing so, we rather draw an ellipse defined by
\[
\mathbf{a}_{k|k}^T \mathbf{X}_{k|k}^{-1} \mathbf{a}_{k|k} = h
\]
(65)
and interpret it directly as the estimated extension as depicted in Fig. 2. In contrast to above, \( h \) here is an arbitrarily chosen yet fixed value. In view of the chosen interpretation of eq. (64), the true extension would be given by
\[
\mathbf{a}_{k|k}^T \mathbf{X}_{k|k}^{-1} \mathbf{a}_{k|k} = h
\]
(66)
Hence, our goal must be to find a region such that one gets with prescribed probability \( c \) an \( \mathbf{X}_k \) where the complete set of points \( \mathbf{a}_k \) obeying, for this \( \mathbf{X}_k \), eq. (65) remains in that region. Herein, we note from Fig. 2 that such an \( \mathbf{X}_k \) can yield both a different orientation as well as a different ratio of semi-axes in comparison with what results from the estimated \( \mathbf{X}_{k|k} \).

At this point, we propose to use as borders of the confidence region two different height lines determined by the estimate as also depicted in Fig. 2. More precisely, we require the probability \( c \) for getting an \( \mathbf{X}_k \) such that
\[
h_{\text{min}} \leq \mathbf{a}_{k|k}^T \mathbf{X}_{k|k}^{-1} \mathbf{a}_{k|k} \leq h_{\text{max}}
\]
(67)
holds for all \( \mathbf{a}_k \) obeying eq. (65). Now, consider any such point \( \mathbf{a}_k \) and compute the corresponding value
\[
\gamma := \mathbf{a}_{k|k}^T \mathbf{X}_{k|k}^{-1} \mathbf{a}_{k|k}
\]
(68)
Then, the choice
\[
\mathbf{a}_{k|k} := \sqrt{\frac{h}{\gamma}} \mathbf{a}_k \iff \mathbf{a}_k = \sqrt{\frac{\gamma}{h}} \mathbf{a}_{k|k}
\]
(69)
with \[ X_{k|k}^{-1/2} (X_{k|k}^{-1/2})^T = X_{k|k} \] (77)
and taking into account that the solutions for \( \lambda \) as determined by eq. (74) are just the eigenvalues of \( Z_k \), we finally obtain from eqs. (72) and (75) the conditions

\[ \lambda_{\text{min}}(Z_k) \geq 1/f \quad \text{and} \quad \lambda_{\text{max}}(Z_k) \leq f \] (78)

being independent of the arbitrarily chosen value \( h \). What we thus need is the joint distribution of the smallest and the largest eigenvalue of \( Z_k \).

Now, if \( X_k \) is distributed according to an inverse Wishart density with expected matrix \( X_{k|k} \) and parameter \( \alpha_{k|k} \), then there holds

\[ X_{k}^{-1} \sim \mathcal{W} \left( \nu_{k|k}, \alpha_{k|k}^{-1} X_{k|k}^{-1} \right) \] (79)
and, consequently,

\[ Z_k \sim \mathcal{W} \left( \nu_{k|k}, \alpha_{k|k}^{-1} \right) \] (80)
possesses a standard Wishart density

\[ Z_k \sim \mathcal{W} \left( \nu_{k|k}, \alpha_{k|k}^{-1} I \right) \] (81)

Herewith, it becomes apparent that the sought mapping between \( c \) and \( f \) does not depend on the estimate \( X_{k|k} \), but solely on the parameter \( \alpha_{k|k} = \nu_{k|k} - d - 1 \).

From [14], we know the joint distribution describing the ordered eigenvalues of a SPD random matrix with standard Wishart density (81). For two spatial dimensions, it is given by

\[ p(\lambda_{\text{max}}, \lambda_{\text{min}}) = \frac{\sqrt{\pi}}{(2 \alpha_{k|k})^{\nu_{k|k}/2} \Gamma(\nu_{k|k}/2) \Gamma(\nu_{k|k} - 1)/2} \] \cdot \exp \left( -\frac{\alpha_{k|k}}{2} (\lambda_{\text{max}} + \lambda_{\text{min}}) \right) \cdot \left( \lambda_{\text{max}} \lambda_{\text{min}} \right)^{(\nu_{k|k} - 3)/2} \cdot (\lambda_{\text{max}} - \lambda_{\text{min}}) \] (82)

for \( \lambda_{\text{max}} \geq \lambda_{\text{min}} \geq 0 \). But, carefully using some partial integration, one can show

\[ \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} e^{-\frac{\lambda_1 + \lambda_2}{2} x^2} (x^{r+1} y^{r+1}) \, dy \, dx \]
\[ = 4 \mu \int_{\lambda_1}^{\lambda_2} e^{-\frac{\lambda_1 x^2}{2}} e^{-\frac{\lambda_2}{2} x^2 + 1} \, dx \]
\[ = 2 \mu \left( e^{-\frac{\lambda_1}{2} x^2 + 1} + e^{-\frac{\lambda_2}{2} x^2 + 1} \right) \int_{\lambda_1}^{\lambda_2} e^{-\frac{\lambda_1}{2} x^2} \, dx \]

to hold and thus obtains, after some more computations,

\[ c(f) = P \left\{ 1/f \leq \lambda_{\text{min}} \land \lambda_{\text{max}} \leq f \right\} \]
\[ = P(2 \beta, 2 \Lambda_2) - P(2(\beta, 2 \Lambda_1)) \]
\[ - \sqrt{\pi} e^{-\Lambda_1 \beta} e^{-\Lambda_2 \beta} \Gamma(\beta + 1/2) \cdot (P(\beta, \Lambda_2) - P(\beta, \Lambda_1)) \] (83)

with the abbreviations

\[ \Lambda_1 := \frac{\alpha_{k|k}}{2f}, \quad \Lambda_2 := \frac{\alpha_{k|k} f}{2}, \quad \beta := \frac{\alpha_{k|k}}{2} + 1 \] (84)
and the normalized incomplete gamma function

\[
P(r, x) = \frac{1}{\Gamma(r)} \int_0^x e^{-\xi r^{-1}} d\xi
\]  

The function \(c(f)\) in eq. (84) obviously is strictly increasing and thus invertible, the sought mapping \(f(c)\) is easily obtained from \(c(f)\) for any given confidence \(c\) by standard means.

7 Simulation results

Using the approach of section 3, we have simulated a formation of 5 individual targets flying with constant speed \(v = 300 \text{ m/s}\) in the \((x, y)\)-plane. The targets were arranged in a line with 500 m distance between neighboring targets where the formation first went through a 45° and two 90° turns (with radial accelerations 2 g, 2 g, and 1 g, respectively) before performing a split-off maneuver. This formation was observed by a (fictitious) sensor with scan time \(T = 10\text{ s}\) delivering uncorrelated noisy \(x\)- and \(y\)-measurements with standard deviations \(\sigma_x = 500\text{ m}\) and \(\sigma_y = 100\text{ m}\), respectively, where we assumed a probability of detection \(P_d = 80\%\) for each individual target (not considering the problem of limited sensor resolution here). With this, the true measurement likelihood is a Gaussian mixture and the assumption (24) can only be a (good or bad) approximation thereof.

To this scenario, we have applied an IMM as described in section 5. The IMM consisted of three white acceleration models for the kinematics where we have combined a low kinematic process noise with a low extension agility, a high kinematic process noise with some very high extension agility—this model accounts for possibly rapid changes of
shape, size, and (absolute) orientation during maneuvers that may be initiated by some lead target—plus a third model with moderate kinematic process noise and high extension agility covering extension changes of formations that do not maneuver too much.

Figure 4. Model probabilities after update for the scenario in Fig. 3 for an IMM consisting of a low-noise model (–), a high-noise model with very high extension agility (−), and a model with moderate process noise and high extension agility (−).

Figure 5. Model-specific values $\alpha_{k|k}^j$ and combined value $\alpha_{k|k}$ for the scenario in Fig. 3.

In Fig. 3, the outcome of a simulation run for this scenario is shown. The implemented algorithm shows both the desired smoothing behavior during non-maneuvering phases as well as some fairly quick responses to maneuvers and/or changes of group extension. Herein, the models are able to follow the change in orientation that is performed by the formation. The depicted confidence regions for the group extent clearly visualize the reduced certainty during and immediately after maneuvers while showing an increasing confidence in those phases where the formation moves with constant velocity and unaltered arrangement.

Fig. 4 shows how the individual model probabilities vary over time. The high-noise model responds to rapid changes of the centroid kinematics. After the maneuvers have finished (as well as in the middle of the more shallow 1 g maneuver), the model adapted to extent changes exhibits some increased contributions before the low-noise model takes over again after adjustment of the orientation. And, the former model is dominant in the split-off phase where the centroid does not show any significant maneuvering behavior but the extent of the formation changes rapidly.

Fig. 5 wraps up this example by plotting the model-specific values $\alpha_{k|k}^j$ as well as the combined value $\alpha_{k|k}$. Obviously, this combined value is mainly driven by the corresponding value of that respective model that is the most dominant. Note, however, that the usage of the scalar moment matching of section 4 does not imply $\alpha_{k|k}$ to be a mere weighted mean of the $\alpha_{k|k}^j$. Because of this, the combined value may well be smaller than those of all of the contributors.

8 Conclusion
Some recently published approaches to tracking of extended objects and group targets delivering more than one measurement per scan model the extent by means of a symmetric positive definite matrix that is assumed to follow an inverse Wishart density. In this paper, some related aspects and properties have been discussed. In context with an IMM, moment matching is a focal issue and we have shown how this can be done for the extension part of the estimation problem. Moreover, we have suggested a heuristic yet practicable way of honoring not only the mean measurement, but also the observed measurement spread when performing the update of the individual model likelihoods. In addition to this, we have proposed and analyzed the usage of a specific confidence region that is related to the estimated ellipsoidal extension.

Still, there is quite some more work to do in the future. This includes the estimation of target numbers within a formation (also accounting for limited sensor resolution by numbers of measurements that depend on object extension) plus, surely the most challenging task, the derivation of sophisticated data association techniques.

References