Exact Algorithms for Four Track-to-Track Fusion Configurations: All you wanted to know but were afraid to ask

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Abstract — In this paper the exact algorithms for track-to-track fusion (T2TF) which can operate at an arbitrary rate for various information configurations are investigated. First, the exact algorithms for T2TF without memory (T2TFwoM) are presented for three information configurations: with no, partial and full information feedback from the fusion center (FC) to the local trackers. As one major contribution of this paper, the impact of information feedback on fusion accuracy is studied. It is shown that in T2TFwoM, which uses only the track estimates at the fusion time, information feedback will have a negative impact on the fusion accuracy. Then, the exact algorithms for T2TF with memory (T2TFwM) are derived for configurations with and without information feedback. Operating at full rate, T2TFwM is equivalent to the centralized measurement fusion (CMF) regardless of information feedback. However, at a reduced rate, a certain amount of degradation in fusion accuracy is unavoidable. In contrast to the case of T2TFwoM, information feedback is beneficial in T2TFwM.

Keywords: Tracking, track-to-track fusion.

1 Introduction

Track-to-track fusion is very important in distributed tracking systems. Compared to the centralized measurement fusion (CMF), it can be performed at a lower rate to reduce the communication requirements on the system. A number of algorithms for T2TF have been proposed in the literature. In this paper, we focus on the exact algorithms for track-to-track fusion (T2TF) at an arbitrary rate under the assumption that all the local track estimates are jointly Gaussian. In [3] the crosscorrelation among tracks of the same target due to common process noises was first observed, and a formula for the calculation of the crosscovariance was also presented. Based on the formula in [3], the algorithm for the one-scan T2TF, i.e., T2TF without memory (T2TFwoM) was studied in [4], which derived the exact algorithm for T2TFwoM without information feedback (T2TFwoMnf) at an arbitrary rate (see also [15]). Another type of algorithm for T2TF — the information matrix fusion (IMF) — was proposed in [17, 13]. Note that, unlike T2TFwoM, IMF belongs to the class of fuser with memory, since it uses track estimates from the previous fusion. Comparisons between T2TFwoMnf in [4] and IMF can be found in [8, 9], where it is shown that, operating at full rate, T2TFwoMnf is not as accurate as IMF (which is equivalent to CMF when the fuser is operating at full rate), and it was concluded that the suboptimality of T2TFwoMnf is because it is optimal only in ML sense.\(^1\) However, IMF is not optimal when the fuser is operating at reduced rate and, as reported in [10], it causes inconsistency and even divergence. A simulation based comparison on existing fusion algorithms can also be found in [16].

In this paper, the exact T2TF algorithms that can operate at an arbitrary rate are investigated for the following information configurations:

1. T2TFwoMnf (T2TFwoM with no information feedback)
2. T2TFwoMpf (T2TFwoM with partial information feedback)
3. T2TFwoMff (T2TFwoM with full information feedback)
4. T2TFwMnf (T2TFwM with no information feedback)

\(^1\)The actual reason that T2TFwoMnf is (slightly) inferior to IMF operating at full rate (when it is algebraically equivalent to CMF, see [1] Sec. 8.6) is the lack of memory of T2TFwoMnf. This is discussed in detail in Sec. 2.4 and Sec. 3.3 in the sequel.
5. T2TFwMff (T2TFwM with full information feedback)\(^2\)

Except for T2TFwoMnf presented in [4], the results for all the other configurations are new. The impact of information feedback and memory on the accuracy of T2TF is thoroughly examined.

The T2TFwoM is known to have a degradation in fusion accuracy compared to CMF. It is shown that information feedback has a negative impact on the fusion accuracy of T2TFwoM. Then the exact algorithms for T2TFwM are derived for configurations with and without information feedback. When operating at full rate, T2TFwM with or without information feedback is equivalent to CMF (which is the global optimum). However, at reduced rate, a certain amount of loss in fusion accuracy is unavoidable, and in contrast to the case of T2TFwoM, information feedback improves the fusion accuracy of T2TFwM. Furthermore, unlike the IMF, the exact T2TFwM derived in this paper is optimal at any rate.

The paper is organized as follows. The exact algorithms and performance evaluations of T2TFwM and T2TFwoM are presented in Sec. 2 and Sec. 3 respectively. Due to space limitation only the case of synchronous sensors is presented, asynchronous sensors will be considered in a separate paper. Sec. 4 summarizes the paper.

2 Exact Algorithms: T2TFwoM at an Arbitrary Rate

This section investigates the exact algorithms for T2TFwoM at an arbitrary rate. Sec. 2.1 formulates the problem. In Sec. 2.2 the exact T2TF algorithms for the three information configurations: with no, partial and full information feedback are presented. In Sec. 2.3, simulation results are presented to compare the fusion accuracies of T2TFwoMnf, T2TFwoMpf, T2TFwMff and CMF. This leads to the observation that, in T2TFwoM, information feedback will cause a degradation of the fusion accuracy. This phenomenon is further explained in Sec. 2.4.

2.1 Problem Formulation: T2TFwoM

For T2TFwoM (configuration II [1]), depending on the existence of information feedback, the three information configurations [1, 14] are illustrated in Fig. 1. Suppose there are two tracks (that pertain to the same target) which are fused at certain times. The first configuration is the T2TFwoM with no information feedback (T2TFwoMnf) [6] corresponding to the Type IIa configuration for multisensor tracking (see Fig. 8.2.3-1 in [1]). As indicated in Fig. 1(a), the two local tracks evolve independently without the information from each other, thus the improved accuracies are achieved only at the fusion times at the FC. The second configuration is the T2TFwoM with partial information feedback (T2TFwoMpf) which belongs to the Type IIb configuration in [1]. In this case, as shown in Fig. 1(b), track 1 is fused with track 2 and continues with the fused track (feedback) from the FC. However, track 2 does not receive the fused track in view of the partial information feedback. The third configuration is the T2TFwoM with full information feedback (T2TFwMff), which also belongs to the Type IIb configuration in Sec. 8.2.3 [1]. As shown in Fig. 1(c), both local trackers receive and continue with the fused track.

Consider the basic scenario with two local trackers (designated as 1 and 2) at different locations. Each tracker obtains measurements with its local sensor and maintains local tracks of the targets. For the sake of simplicity, it is assumed that all the sensors are synchronized and obtain measurements with sampling interval \(T\). Communication links are available between the FC and the local trackers. Each local tracker is allowed to communicate with the FC once in a while, sending its tracks to the FC and possibly receiving the fused tracks (when there is information feedback). At the FC, the fusion (without memory) of the tracks of the target from trackers 1 and 2 is formulated as follows.\(^3\)

\[^2\]The T2TFwoMpf (T2TFwM with partial information feedback), which is a minor modification of T2TFwMff, is omitted for conciseness.

\[^3\]Association (see [1] Sec. 8.4) is assumed to have been already performed.
Let \( \hat{x}_1(k|k), P_1(k|k) \) and \( \hat{x}_2(k|k), P_2(k|k) \) represent the two local tracks at the fusion time. Assuming, for now, the crosscovariance of the two tracks \( P_{12}(k|k) \) is available at the FC, T2TFwoM should be performed, so that

\[
[\hat{x}_c(k|k), P_c(k|k)] = \Phi [\hat{x}_1(k|k), P_1(k|k), \hat{x}_2(k|k), P_2(k|k), P_{12}(k|k)] \tag{1}
\]

where \( [\hat{x}_1(k|k), P_1(k|k), \hat{x}_2(k|k), P_2(k|k), P_{12}(k|k)] \) represent the fused track. After T2TF, the local tracks and their crosscovariance should also be updated to \( \hat{x}_1^2(k|k), P_{11}(k|k), \hat{x}_2(k|k), P_{22}(k|k) \) and \( P_{12}(k|k) \) according to the information configuration of the fusion. Throughout the paper, superscript "s" is used to indicate post-fusion tracks. Note that (1) implies that only the local track estimates at the fusion time are used for T2TFwoM, i.e., this is a fuser without memory of fused and local track estimates from the previous fusion.

### 2.2 The Exact T2TFwoM Algorithms

If the tracks \( \hat{x}_1(k|k), P_1(k|k), \hat{x}_2(k|k), P_2(k|k) \) and their crosscovariance \( P_{12}(k|k) \) are available at the FC, the optimal\(^4\) T2TFwoM can be done according to Eqs. (8.4.4-4)–(8.4.4-5) in [1], namely,

\[
\hat{x}_c(k|k) = \hat{x}_1(k|k) + \left[ P_1(k|k) + P_2(k|k) - P_{12}(k|k) - P_{21}(k|k) \right]^{-1} \cdot \left[ \hat{x}_1(k|k) - \hat{x}_2^2(k|k) \hat{x}_2(k|k) - \hat{x}_1(k|k) \right] \tag{2}
\]

\[
P_c(k|k) = P_1(k|k) - \left[ P_1(k|k) + P_2(k|k) - P_{12}(k|k) - P_{21}(k|k) \right]^{-1} \cdot \left[ P_2(k|k) - P_{21}(k|k) \right] \tag{3}
\]

where

\[
P_{12}(k|k) = P_{21}(k|k) = \text{Cov}[^T \hat{x}_1(k|k), \hat{x}_2(k|k)] \tag{4}
\]

To calculate \( P_{12}(k|k) \), suppose the previous fusion was performed at discretized time \( l \), after which one has the errors

\[
\tilde{x}_1^2(l|l) = \tilde{x}_1^2(l|l) - x(l) \tag{5}
\]

\[
\tilde{x}_2^2(l|l) = \tilde{x}_2^2(l|l) - x(l) \tag{6}
\]

where \( x(l) \) denotes the true state of the target at \( l \). Let

\[
P_{11}^* (l|l) = \text{Cov}[^T \tilde{x}_1^2(l|l), \tilde{x}_1^2(l|l)] \tag{7}
\]

\[
P_{22}^* (l|l) = \text{Cov}[^T \tilde{x}_2^2(l|l), \tilde{x}_2^2(l|l)] \tag{8}
\]

\[
P_{12}^* (l|l) = \text{Cov}[^T \tilde{x}_1^2(l|l), \tilde{x}_2^2(l|l)] \tag{9}
\]

From Eq. (8.4.2-2) in [1], one has

\[
\tilde{x}_s(l+1|l+1) = [I - K_s(l+1)H_s(l+1)]F(l)\tilde{x}_s(l|l) - [I - K_s(l+1)H_s(l+1)]v(l) + K_s(l+1)v_s(l|l), \quad s = 1, 2 \tag{10}
\]

where \( K_s(\cdot) \) is the Kalman filter gain; \( H_s(\cdot) \) is the observation matrix and \( w_s(\cdot) \) is the measurement noise at local tracker \( s \); \( F(\cdot) \) is the state transition matrix and \( v(\cdot) \) is the process noise. Using (10) recursively for both local tracks from discrete time \( l \) to \( k \), it follows that

\[
\tilde{x}_s(k|k) = W_{s}^w(k, l) \tilde{x}_s^*(l|l) + \sum_{i=l+1}^{k} W_{s}^w(k, i-1) v(i-1)
\]

\[
+ \sum_{i=l+1}^{k} W_{s}^w(k, i) w_s(i), \quad s = 1, 2 \tag{11}
\]

where the weights are defined as

\[
W_{s}^w(k, l) = \prod_{i=0}^{k-l-1} [I - K_s(k-i)H_s(k-i)]F(k-i-1) \tag{12}
\]

\[
W_{s}^w(k, i-1) = - \left\{ \prod_{j=0}^{k-i-1} [I - K_s(k-j)H_s(k-j)]F(k-j-1) \right\} \cdot [I - K_s(i)H_s(i)] \tag{13}
\]

\[
W_{s}^w(k, i) = \left\{ \prod_{j=0}^{k-i-1} [I - K_s(k-j)H_s(k-j)]F(k-j-1) \right\} \cdot K_s(i) \tag{14}
\]

Eq. (11) is the expression of the errors of the tracks from Kalman filters as weighted sums of the error at a certain point and the intervening process and measurement noises. The significance of this expression is that it shows explicitly all the sources of uncertainty and provides the general tool for the derivations of the T2TF algorithms in the absence or presence of memory and feedback.

From (11) and the whiteness assumption of the measurement noises and the process noises, the crosscovariance \( P_{12}(k|k) \), required by the exact T2TFwoM given in (2)–(3), can be calculated as

\[
P_{12}(k|k) = W_{11}^w (k, l) P_{12}^w (l|l) W_{22}^w (k, l)'
\]

\[
+ \sum_{i=l+1}^{k} W_{1}^w(k, i-1) Q(i-1) W_{2}^w(k, i-1) \tag{15}
\]

where \( Q(\cdot) \) is the covariance of the process noise. Note that (15) requires all the local filter gains and observation matrices since the last fusion time. A method for the approximate calculation of (15) at the FC was proposed in [18], which has much less communication requirements and practically no loss in fusion accuracy.

Similarly to (11), the error of the fused track (2) can be expressed as

\[
\tilde{x}_c(k|k) = \hat{x}_1^2(k|k) + K_{12}(k)[\hat{x}_2^2(k|k) - \hat{x}_1^2(k|k)]
\]

\[
= [I - K_{12}(k)]\hat{x}_1^2(k|k) + K_{12}(k)\hat{x}_2^2(k|k) \tag{16}
\]
After T2TF, local tracks 1 and 2 and their crosscovariance should be updated according to the information configuration of the fusion.

In configuration T2TFwoM (see Fig. 1(a)), one has

$$\dot{x}_1^*(k|k) = \dot{x}_1(k|k)$$  
$$P_1^*(k|k) = P_1(k|k)$$  
$$\dot{x}_2^*(k|k) = \dot{x}_2(k|k)$$  
$$P_2^*(k|k) = P_2(k|k)$$  
$$P_{12}^*(k|k) = P_{12}(k|k)$$

where $P_{12}(k|k)$ is given in (15).

In configuration T2TFwoMpf (see Fig. 1(b))

$$\dot{x}_1^*(k|k) = \dot{x}_e(k|k)$$  
$$P_1^*(k|k) = P_e(k|k)$$  
$$\dot{x}_2^*(k|k) = \dot{x}_e(k|k)$$  
$$P_2^*(k|k) = P_e(k|k)$$  
$$P_{12}^*(k|k) = P_e(k|k)$$

and according to (16)

$$P_{12}^*(k|k) = [I - K_{12}(k)]P_{12}(k|k) + K_{12}(k)P_2(k|k)$$

In configuration T2TFwoMff (see Fig. 1(c))

$$\dot{x}_2^*(k|k) = \dot{x}_1^*(k|k) = \dot{x}_e(k|k)$$  
$$P_2^*(k|k) = P_1^*(k|k) = P_e(k|k)$$  
$$P_{12}^*(k|k) = P_e(k|k)$$

The exact algorithm of T2TFwoM is summarized as follows:

- At the FC, the local tracks are fused according to (2)–(3).
- T2TFwoM can be done exactly, if the following data are available:
  (i) The local tracks to be fused: $\dot{x}_1(k|k)$, $P_1(k|k)$ and $\dot{x}_2(k|k)$, $P_2(k|k)$
  (ii) The covariances and crosscovariance from the previous fusion at time $l$: $P_{11}^*(l|l)$, $P_{22}^*(l|l)$, $P_{12}^*(l|l)$ (see (7)–(8)) — needed for the calculation of the current crosscovariance.
  (iii) The local weights (12)–(13).
- Depending on the information configuration, the local tracks are updated using (17)–(21) for T2TFwoMnf, or (22)–(26) for T2TFwoMpf, or (27)–(29) for T2TFwoMff.

The exact algorithm of T2TFwoM has no theoretical limit on the number of the local trackers. Only the crosscovariances among all the tracks of the same target need to be properly calculated. See [11] for the $n$-sensors version of the fusion equations (2)–(3). The use of the results from [11] in the general case requires Eqs. (11)–(13) for each sensor and (15) for each pair of sensors.

### 2.3 Comparison of the Exact T2TFwoM Algorithms and CMF

The exact algorithms for T2TFwoM are evaluated first in the following tracking scenario. The target state is defined as $[x\ x^\prime]^T$. The target motion is modeled as the DWNA model in [2], Sec. 6.3.2. It is assumed that two sensors obtain position measurements of the target with a sampling interval of $T = 1s$. The standard deviation of the measurement noise is $\sigma_w = 30m$ and the process noise variance is $q = 1m^2/s^4$. T2TFwoM takes place every 5 s, i.e., at a reduced rate.

<table>
<thead>
<tr>
<th>Fusion Type</th>
<th>FC track at fusion time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pos</td>
</tr>
<tr>
<td>T2TFwoMff</td>
<td>133</td>
</tr>
<tr>
<td>T2TFwoMpf</td>
<td>131</td>
</tr>
<tr>
<td>T2TFwoMnf</td>
<td>125</td>
</tr>
<tr>
<td>CMF</td>
<td>119</td>
</tr>
</tbody>
</table>

All the fusers are consistent in the simulations (their covariance calculations are exact). In view of their consistency, the performance comparison can be made using the fuser-calculated covariances. Table 1 shows the steady state variances of position and velocity at the FC. All the fused tracks are more accurate than the single-sensor (local) tracks without fusion, which have steady state variances as 205 in position and 7.26 in velocity. Note that at the fusion time the position estimates of all the fused tracks have a small degradation compared to the CMF: 5% for T2TFwoMnf, 10% for T2TFwoMpf, 12% for T2TFwoMff. This shows that T2TFwoM has a degradation in fusion accuracy compared to the CMF and the degradation increases in the presence of information feedback. This apparently counterintuitive result is further discussed in the next subsection.

### 2.4 The Impact of Information Feedback on T2TFwoM

To show the impact of information feedback on T2TFwoM, consider the following scalar estimation problem. At time 0, two local estimators have independent prior information: $\bar{x}_1 \sim N(x(0), P_1)$ at estimator 1 and $\bar{x}_2 \sim N(x(0), P_2)$ at estimator 2. At time 1, $x(1) = x(0) + v_1$ where the process noise $v_1 \sim N(0, Q)$. The estimators have independent measurements of the state: $z_1 = x(1) + w_1$ ($w_1 \sim N(0, R_1)$) at estimator 1 and $z_2 = x(1) + w_2$ ($w_2 \sim N(0, R_2)$) at estimator 2. The errors in the prior information, the process noise and the measurement noises are all independent. For the sake of simplicity it is assumed that $P_1 = P_2 = R_1 = R_2 = R = 1$ and $Q = 1/2$. T2TFwoM
occurs at time 0 and 1. Consider the fusion at time 1.

\[
\begin{align*}
\bar{x}^{\mathrm{CMF}}(1|1) &= \frac{1}{6} \hat{x}_1 + \frac{1}{6} \hat{x}_2 + \frac{1}{3} \bar{z}_1 + \frac{1}{3} \bar{z}_2 \\
\end{align*}
\] (30)

with error (using the weighted sum form)

\[
\begin{align*}
\hat{x}^{\mathrm{CMF}}(1|1) &= \frac{1}{6} \hat{x}_1 + \frac{1}{6} \hat{x}_2 - \frac{1}{3} v_1 + \frac{1}{3} w_1 + \frac{1}{3} w_2 \\
\end{align*}
\] (31)

where \( \hat{x}_1 \) and \( \hat{x}_2 \) denote the errors of \( \bar{x}_1 \) and \( \bar{x}_2 \). It is easy to calculate the covariance

\[
\text{Cov}[\hat{x}^{\mathrm{CMF}}(1|1)] = \frac{1}{3} 
\] (32)

In T2TFwoMff6, one has

\[
\begin{align*}
\hat{x}^{\mathrm{ff}}(1|1) &= \frac{1}{4} \hat{x}_1 + \frac{1}{4} \hat{x}_2 + \frac{1}{2} \bar{z}_1 + \frac{1}{2} \bar{z}_2 \\
\end{align*}
\] (33)

with error

\[
\begin{align*}
\hat{x}^{\mathrm{ff}}(1|1) &= \frac{1}{4} \hat{x}_1 + \frac{1}{4} \hat{x}_2 - \frac{1}{2} v_1 + \frac{1}{2} w_1 + \frac{1}{2} w_2 \\
\end{align*}
\] (34)

\[
\text{Cov}[\hat{x}^{\mathrm{ff}}(1|1)] = \frac{3}{8} 
\] (35)

In T2TFwoMnf, it follows that

\[
\begin{align*}
\hat{x}^{\mathrm{nf}}(1|1) &= \frac{1}{5} \hat{x}_1 + \frac{1}{5} \hat{x}_2 + \frac{3}{10} z_1 + \frac{3}{10} z_2 \\
\end{align*}
\] (36)

with error

\[
\begin{align*}
\hat{x}^{\mathrm{nf}}(1|1) &= \frac{1}{5} \hat{x}_1 + \frac{1}{5} \hat{x}_2 - \frac{2}{5} v_1 + \frac{3}{10} w_1 + \frac{3}{10} w_2 \\
\end{align*}
\] (37)

and

\[
\text{Cov}[\hat{x}^{\mathrm{nf}}(1|1)] = \frac{17}{50} 
\] (38)

It can be seen

\[
\text{Cov}[\hat{x}^{\mathrm{CMF}}(1|1)] < \text{Cov}[\hat{x}^{\mathrm{nf}}(1|1)] < \text{Cov}[\hat{x}^{\mathrm{ff}}(1|1)] 
\] (39)

There are losses in accuracy in T2TFwoMff and T2TFwoMnf compared to CMF, although they are relatively small, due to the large process noise in the example. Comparing (31), (34) and (37), it can be seen that the weights of the measurements are lower in T2TFwoMnf than in CMF. They become even lower in T2TFwoM due to the information feedback, which leads to the further loss in fusion accuracy.

### 3 Exact T2TFwM Algorithms for Arbitrary Rate

The results in Sec. 2.4 show that, at any rate, T2TFwoM is less accurate than CMF and information feedback is detrimental to T2TFwoM. However, IMF [13] is equivalent to CMF when operating at full rate. In this case, T2TFwoM is inferior to IMF. This is because T2TFwoM uses only local estimates at the fusion time, which contain most but not all of the information for T2TF. In contrast, IMF belongs to the class of T2TFwM.

To account for the information from the fused and local track estimates from the previous fusion, the exact algorithm for T2TFwM at an arbitrary rate is derived in the next two subsections for configurations with no information feedback (T2TFwoMnf) and full information feedback (T2TFwoMff). Fig. 3 shows the information flow of these two configurations.

#### 3.1 The Exact T2TFwMnf Algorithm

As shown in Fig. 3(a), for T2TFwoMnf, at fusion time \( k \), the track estimates to be fused are local track estimates \( \hat{x}_1(k|k) \), \( \hat{x}_2(k|k) \) and the predicted track estimates \( \hat{x}_1(k|l) \), \( \hat{x}_2(k|l) \) and \( \hat{x}_c(k|l) \) from the previous fusion, where subscript “c” indicates the (fused) track at the FC. Stacking the estimates and the predicted estimates as a vector, one has

\[
\mu = [\hat{x}_1(k|k) \; \hat{x}_2(k|k) \; \hat{x}_c(k|l) \; \hat{x}_1(k|l) \; \hat{x}_2(k|l)]^T 
\] (40)
\[ \dot{x}(k|k) = \dot{x}(k|l) + \text{Cov}[x(k), \nu] \text{Cov}(\nu)^{-1} \nu \] (46)

where \( x(k) \) is the true state of the target at time \( k \), and

\[ \text{Cov}[x(k), \nu] = -[\text{Cov}(\mu)]_{(3,3)} M' \] (47)

\[ \text{Cov}(\nu) = M \text{Cov}(\mu) M' \] (48)

where, \([\text{Cov}(\mu)]_{(i,:)}\) denotes the \( i \)th row of matrix \( \text{Cov}(\mu) \) and the 3rd row \(-[\text{Cov}(\mu)]_{(3,:)}\) corresponds to \( \dot{x}_c(k|l) \). In (46) \( \dot{x}_c(k|l) \) plays the role of the prior at the FC and the other elements of \( \mu \) play the role of the observations.

From (45)–(48), the fused estimate is

\[ \dot{x}_c(k|k) = \dot{x}_c(k|l) - [\text{Cov}(\mu)]_{(3,:)} M'(M \text{Cov}(\mu) M')^{-1} M \mu \] (49)

where

\[ K_\mu \triangleq -[\text{Cov}(\mu)]_{(3,:)} M'(M \text{Cov}(\mu) M')^{-1} M \] (50)

The fused covariance is

\[ P_c(k|k) = P_c(k|l) - \text{Cov}[x(k), \nu] \text{Cov}(\nu)^{-1} \text{Cov}[x(k), \nu]' \]

\[ = P_c(k|l) - [\text{Cov}(\mu)]_{(3,:)} M'(M \text{Cov}(\mu) M')^{-1} \]

\[ \cdot M' \left[ [\text{Cov}(\mu)]_{(3,:)} \right]' \]

\[ = P_c(k|l) + K_\mu \left[ [\text{Cov}(\mu)]_{(3,:)} \right]' \] (51)

For T2TFwMnf the crosscovariances between the fused track and the tracks from tracker 1 and 2 can be obtained from (49) as

\[ \text{Cov}[\dot{x}_1(k|k), \dot{x}_c(k|k)] = [\text{Cov}(\mu)]_{(1,3)} + [\text{Cov}(\mu)]_{(1,:)} K_\mu' \] (52)

\[ \text{Cov}[\dot{x}_2(k|k), \dot{x}_c(k|k)] = [\text{Cov}(\mu)]_{(2,3)} + [\text{Cov}(\mu)]_{(2,:)} K_\mu' \] (53)

where \([\text{Cov}(\mu)]_{(i,j)}\) is element \((i, j)\) of \( \text{Cov}(\mu) \).

### 3.2 The Exact T2TFwMff Algorithm

In contrast to T2TFwMnf, in T2TFwMff, one has \( \dot{x}_1(k|l) = \dot{x}_2(k|l) = \dot{x}_c(k|l) \). Consequently, the elements \( \dot{x}_1(k|l) \) and \( \dot{x}_2(k|l) \) in (40) should be removed. In this case, redefine \( \mu \) in (40) and \( M \) in (44) as

\[ \mu = [\dot{x}_1(k|k) \; \dot{x}_2(k|k) \; \dot{x}_c(k|l)]' \] (54)

and

\[ M = \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \] (55)

It follows that

\[ \nu = \begin{bmatrix} \dot{x}_1(k|k) - \dot{x}_c(k|l) \\ \dot{x}_2(k|k) - \dot{x}_c(k|l) \\ \dot{x}_c(k|l) \end{bmatrix} = M \mu \] (56)
Then, similarly to T2TFwMnf, the fused estimate is obtained using (49) and the fused covariance follows from (51) using the modified definitions (54) and (55).

Note that, in T2TFwMff the crosscovariances between the fused track and the local tracks after the information feedback are the same with the fused covariance (51). This is different from the updated crosscovariances in T2TFwMnf, i.e., (52) and (53).

### 3.3 T2TFwMnf vs. T2TFwMff, CMF

To evaluate the performance of the optimal T2TF with memory (T2TFwM) at an arbitrary rate, consider the following tracking scenario. The state of the target (taken as a scalar for simplicity) evolves according to

\[ x(k) = x(k-1) + v(k) \quad k = 2, 3, \ldots \]  

where \( v(k) \) is the process noise with variance \( q \).

There are two trackers, 1 and 2, taking measurements of the target with measurement noises \( w_1 \) and \( w_2 \), namely,

\[ z_i(k) = x(k) + w_i(k) \quad i = 1, 2 \]

where \( w_i(k) \) are zero-mean Gaussian noises with variance \( R_i \). The two trackers calculate tracks of the target with their own measurements using a Kalman filter. Each local track is initialized at time 1 with the first local measurement. The first T2TF happens at time 1. Then T2TFwM occurs every \( N_f \) sampling times.

Table 2: Fuser calculated variances at fusion times for \( N_f = 1 \) (full rate), \( q = 0.3 \), \( R_1 = R_2 = 1 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>T2TFwMnf Tracker1</td>
<td>1.00</td>
<td>0.565</td>
<td>0.464</td>
<td>0.433</td>
</tr>
<tr>
<td>Fuser</td>
<td>0.50</td>
<td>0.307</td>
<td>0.274</td>
<td>0.267</td>
</tr>
<tr>
<td>T2TFwMff Fuser</td>
<td>0.50</td>
<td>0.307</td>
<td>0.274</td>
<td>0.267</td>
</tr>
<tr>
<td>CMF</td>
<td>0.50</td>
<td>0.307</td>
<td>0.274</td>
<td>0.267</td>
</tr>
</tbody>
</table>

Table 3: Fuser calculated variances at fusion times for \( N_f = 3 \) (reduced rate), \( q = 0.3 \), \( R_1 = R_2 = 1 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>T2TFwMnf Tracker1</td>
<td>1.00</td>
<td>0.464</td>
<td>0.420</td>
<td>0.418</td>
</tr>
<tr>
<td>Fuser</td>
<td>0.50</td>
<td>0.277</td>
<td>0.270</td>
<td>0.269</td>
</tr>
<tr>
<td>T2TFwMff Fuser</td>
<td>0.50</td>
<td>0.276</td>
<td>0.268</td>
<td>0.268</td>
</tr>
<tr>
<td>CMF</td>
<td>0.50</td>
<td>0.274</td>
<td>0.265</td>
<td>0.265</td>
</tr>
</tbody>
</table>

Table 2 shows the fuser- and tracker-calculated variances of the errors of the track estimates when the fuser is operating at full rate. It can be seen that, at full rate, the exact fuser with memory (with or without information feedback) is equivalent to the CMF.

Table 3 shows the fuser- and tracker-calculated variances when the fuser (with memory) is operating at reduced rate. In this case, T2TFwM, with or without information feedback, has a small loss in fusion accuracy compared to the CMF. This is because, in T2TF, information from the common process noises and the common prior information (due to information feedback) appears simultaneously in different local tracks, which causes their weights (and, consequently, also the weights of the new measurements) in the fused track to deviate from the global optimum. This is similar to the T2TFwoM example discussed in Sec. 2.4. When T2TFwM is done at full rate, these deviations are fully corrected by fusing also the previous track estimates (see [1], Sec. 8.6). However, at a lower rate, the deviations can not be fully corrected, thus, a certain amount of degradation in fusion accuracy is unavoidable. Also note that, in contrast to the case of T2TFwoM, at a reduced rate T2TFwMff is more accurate than T2TFwMnf, namely information feedback improves fusion accuracy in T2TFwM (as expected). While it is too involved to provide a theoretical proof of this result, simulations in different settings confirm it.

IMF which also uses the previous track estimates (i.e., it has memory), when operating at full rate, is algebraically equivalent to the CMF (see [1], Sec. 8.6) and also to the exact algorithms for T2TFwM presented in this section. However, at a lower rate, IMF is not an exact algorithm. As reported in [10], this causes inconsistency and may even lead to divergence. In contrast, the exact algorithms for T2TFwMnf and T2TFwMff are optimal at any rate.

### 4 Conclusions

In this paper the exact Track-to-Track fusion (T2TF) algorithms at an arbitrary rate are investigated for various information configurations. First the exact algorithms for T2TF without memory (T2TFwM: fuser uses only the local track estimates at the fusion time) are presented for three information configurations, namely, T2TFwM with no, partial and full information feedback. It is shown that, for T2TFwoM, information feedback is detrimental to fusion accuracy. Then the exact algorithms for T2TF with memory (T2TFwM: fuser uses also the fused and local track estimates from the previous fusion) at an arbitrary rate are derived for information configurations with and without information feedback. At full rate, T2TFwM, with or without information feedback, is equivalent to the centralized measurement fusion (CMF, which is the global optimum). However, when operating at a lower rate, a certain amount of loss in fusion accuracy (compared to the CMF) is unavoidable. In contrast to the case of
information feedback improves the fusion accuracy of T2TFwM. And, unlike the information matrix fusion (IMF) which is optimal (same as CMF) only at full rate, the exact algorithm T2TFwM is optimal at any rate, although its performance is unavoidably not as good as CMF.

References


