Heavy-Tailed Exponential Distribution: Basic Properties and Parameter Estimation

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Abstract - Heavy-tailed exponential distribution is a direct and necessary extension of the heavy-tailed Rayleigh distribution. First, some basic properties of heavy-tailed exponential distribution are introduced in this paper including the series form of the density function, the heavy-tailed property and the non-existence of finite variance. Second, ratio estimator, logarithmic moment estimator and iterative logarithmic moment estimator are presented to estimate the parameters of heavy-tailed exponential distribution based on negative-order moments. The logarithmic moment estimator with explicit closed form is only determined by samples, and the iterative logarithmic moment estimator achieves better performance only using fewer samples in each step computation. Monte Carlo simulation results demonstrate the high efficiency of the iterative logarithmic moment estimator for heavy-tailed exponential distribution.

Keywords: Heavy-tailed exponential distribution, negative-order moments, logarithmic moment estimation, iterative logarithmic moment estimation.

1 Introduction

Based on the generalized central limit theorem, alpha-stable distribution is the only limiting distribution for the sum of independent and identically distributed random variables, and it is widely used for the modeling of impulsive signal and noise [1]. For synthetic aperture radar (SAR) images, a large number of scatterers are usually assumed to exist in each resolution cell and they are not distinguished from each other, so the total echo in each resolution cell is the sum of all the scattered signals [2]. Therefore, it is a reasonable choice to model the echo as the alpha-stable distribution [1]. Further, assumptions of the symmetric alpha-stable distribution for the real and imagery parts of echo lead to the heavy-tailed Rayleigh distribution for SAR amplitude images [1, 3].

Heavy-tailed Rayleigh distribution is demonstrated to be a generalization of the classical Rayleigh distribution. With the heavier tails than the Rayleigh distribution, the heavy-tailed Rayleigh distribution is a useful tool for the modeling of high resolution SAR amplitude images such as urban scenes and sea surface [3]. In this paper, we generalize the heavy-tailed distribution to the SAR intensity images and we call this distribution as heavy-tailed exponential distribution because of the heavy-tailed property. First, we introduce some basic properties of heavy-tailed exponential distribution such as the series form of density function, the heavy-tailed property and the non-existence of finite variance. Second, we present three methods to estimate the parameters of heavy-tailed exponential distribution based on negative-order moments. We compare the performance of these estimators and demonstrate the highest efficiency of the iterative logarithmic moment estimator according to Monte Carlo simulations.

2 Basic properties of heavy-tailed exponential distribution

It has been demonstrated that the density function of heavy-tailed exponential distribution for SAR intensity images can be written as

\[ f(x) = \frac{1}{2} \int_0^\infty \rho \exp(-\rho x^\alpha) J_0(\rho \sqrt{x}) d\rho, \]  

where \(0 < \alpha \leq 2\) is the characteristic exponent and \(\gamma > 0\) is the scale parameter [4]. Here, \(J_0\) is the zero-order Bessel function of the first kind. Heavy-tailed exponential distribution reduces to the classical exponential distribution when \(\alpha = 2\), which indicates that the exponential distribution is just a special case of the heavy-tailed exponential distribution. Except for the special case of \(\alpha = 1\) and \(\alpha = 2\), no closed-form expressions exist for the general heavy-tailed exponential distribution.

However, using the series form of heavy-tailed Rayleigh density function introduced in [1], we can easily obtain the following series for heavy-tailed exponential density function

\[
\begin{cases}
\frac{1}{\alpha \rho^{\alpha+1}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha k/2+1) \Gamma\left(1+k\alpha/2\right)}{k!} \sin\left(k\alpha x/2\right) x^{\alpha k-1}, & 0 < \alpha < 1, \\
\frac{1}{\alpha \rho^{\alpha+1}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+\alpha)}{k! \Gamma\left(2k+1\right)} \left(\frac{2k+\alpha}{\alpha}\right)^{k} x^{k+\alpha/2}, & 1 < \alpha < 2.
\end{cases}
\]
It has been demonstrated that heavy-tailed Rayleigh distribution has algebraic tails compared to the classical Rayleigh distribution, i.e.,
\[ \lim_{a \to \infty} a^\alpha P(A > a) = (2\pi/\alpha)B(\alpha, \gamma), \quad (3) \]
where \( A \) is a heavy-tailed Rayleigh distributed random variable and \( B(\alpha, \gamma) \) is a positive constant depending on \( \alpha \) and \( \gamma \) [1]. For a heavy-tailed exponential distributed random variable \( I \), making a change of variables \( a = \sqrt{t} \), we can obtain
\[ \lim_{i \to \infty} P(I > i) = \lim_{a \to \infty} a^\alpha P(A > a) = D(\alpha, \gamma). \quad (4) \]
where \( D(\alpha, \gamma) = (\pi/\alpha)B(2\alpha, \gamma) \). Thus, the heavy-tailed exponential distribution also has algebraic tails. This implies that the tails of the heavy-tailed exponential distribution are significantly heavier than those of the traditional exponential distribution, which is just the reason we call this distribution heavy-tailed. The tail of heavy-tailed exponential distribution is plotted in Fig. 1. Obviously, the smaller the value of \( \alpha \) is, the heavier tail the distribution has. In the special case of \( \alpha = 2 \), the distribution has the thinnest tail because the heavy-tailed exponential distribution reduces to the exponential distribution.

![Fig. 1 The tail of heavy-tailed exponential distribution (\( \gamma = 1 \))](image_url)

Similar to the case of symmetric alpha-stable distribution [1], the \( p \) th order moment of the heavy-tailed exponential distribution can be written as
\[ E(I^p) = \rho^p \int_0^\infty \frac{1}{\Gamma(p+1)} \rho \exp(-\rho \gamma) J_0(\rho \sqrt{x}) d\rho dx. \quad (7) \]
After changing the order of integration in (7), making a change of variables \( t = \rho \sqrt{x} \) and some algebraic manipulation, we have
\[ E(I^p) = \int_0^\infty \rho^{2p-1} \exp(-\rho \gamma) d\rho \int_0^\infty t^{p+1} J_0(t) dt . \quad (8) \]
Using the identity introduced in [3], we can obtain the following negative-order moment for heavy-tailed exponential distribution
\[ E(I^p) = \frac{2^{p+1} \Gamma(p+1) \gamma^{p/\alpha} \Gamma(-2p/\alpha)}{\Gamma(-p)} , \quad -1 < p < -1/4 . \quad (9) \]

### 3 Parameter estimation of heavy-tailed exponential distribution

Parameter estimation is necessary for the practical use of heavy-tailed exponential distribution. In this section, we present three methods to estimate the parameters of heavy-tailed exponential distribution based on negative-order moments.

#### 3.1 Negative-order moment of heavy-tailed exponential distribution

If \( X \) is a heavy-tailed exponential distributed random variable, its \( p \) th order moment can be written as
\[ E(X^p) = \int_0^\infty x^p \frac{1}{\Gamma(p+1)} \rho \exp(-\rho \gamma) J_0(\rho \sqrt{x}) d\rho \Gamma(-p) dx . \quad (10) \]
Using the ratio estimator (10), \( E(I^p) \) can be used to test whether a distribution is heavy-tailed exponential distribution or traditional exponential distribution although the value of \( \alpha \) is not known.

#### 3.2 Ratio estimator

If \( I \) is a heavy-tailed exponential distributed random variable, its \( p \) th order moment satisfies (9). We can take the ratio
\[ \frac{E(I^p)}{E^2(I^p)} = \frac{\Gamma(2p+1) \gamma^{2p/\alpha} \Gamma(-2p/\alpha)}{\Gamma(-p) \Gamma(-p-1) \gamma^{2p/\alpha}} , \quad -1 < p < -1/4 . \quad (10) \]
Obviously, \( \alpha \) can be numerically evaluated from (10), and \( \gamma \) can be estimated from (9) as follows
\[ \gamma = \left[ \frac{\alpha \Gamma(-p) E(I^p)}{2^{2p+1} \Gamma(p+1) \Gamma(-2p/\alpha)} \right]^{1/2p} . \quad (11) \]

In the ratio estimator (10) and (11), \( E(I^p) \) can be estimated by the empirical moments calculated from the samples. Table 1 illustrates the average and standard deviation values (in parentheses) of Monte Carlo simulation results based on the ratio estimator. Various numbers of samples from a standard (\( \gamma = 1 \)) heavy-tailed exponential distribution were generated using the method introduced in [5] and the experiment was repeated 100 times independently. Obviously, better performance is
achieved with larger values of $p$. However, it must be stressed that the performance of ratio estimator is degraded seriously when $p$ is chosen inappropriately (e.g., $p = -0.35$).

Table 1 Performance of ratio estimator (true $\alpha = 0.5$, $\gamma = 1$)

<table>
<thead>
<tr>
<th>Number of Samples</th>
<th>$p = -0.35$</th>
<th>$p = -0.3$</th>
<th>$p = -0.26$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\gamma}$</td>
<td>$\hat{\gamma}$</td>
<td>$\hat{\gamma}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.5417 (0.0656)</td>
<td>0.5164 (0.0507)</td>
<td>0.5083 (0.0365)</td>
</tr>
<tr>
<td>2000</td>
<td>0.5356 (0.0529)</td>
<td>0.5099 (0.0443)</td>
<td>0.5027 (0.0287)</td>
</tr>
<tr>
<td>5000</td>
<td>0.5187 (0.0572)</td>
<td>0.5107 (0.0253)</td>
<td>0.5025 (0.0235)</td>
</tr>
</tbody>
</table>

3.3 Logarithmic moment estimator

Performance of the ratio estimator greatly relies on the choice of $p$, which, of course, is not convenient for us [1,6]. Denoting $I$ as a heavy-tailed exponential distributed random variable, we can rewrite the negative-order moment $E(Y^p)$ as $E(e^{p \log(I)})$ and define a new random variable $Y = \log(I)$. Then, expanding $e^{p \log(I)}$ into the Taylor series, we can write $E(e^{p \log(I)})$ as

$$E(e^{p \log(I)}) = \sum_{k=0}^{\infty} E(Y^k) \frac{2^p}{k!} D(p, \alpha) \gamma^\alpha, \quad (12)$$

where $D(p, \alpha) = \frac{2^{p+1} \Gamma(p+1) \Gamma(-2p/\alpha)}{\Gamma(-p)}$. Then, moments of $Y$ of any order can be obtained by

$$E(Y^k) = \frac{d^k \left(D(p, \alpha) \gamma^{2\alpha/\alpha}\right)}{d p^k} \bigg|_{p=0} \quad (13)$$

Considering the first-order and second-order moments of $Y$, after some manipulation, we can obtain

$$E(Y) = 2C_\gamma \left(\frac{1}{\alpha - 1}\right) + \frac{2 \log \gamma}{\alpha} + 2 \log 2 \quad (14)$$

and

$$\text{Var}(Y) = \frac{2\pi^2}{3\alpha^2}. \quad (15)$$

Here, $C_\gamma$ denotes the Euler’s constant. Since $\alpha$ is isolated in (15), it can be easily obtained by $\text{Var}(Y)$, and $\gamma$ can be estimated from (14) immediately.

In (14) and (15), mean and variance of $Y$ can be estimated from the sample mean and the sample variance, respectively. Compared to the ratio estimator (10) and (11), the logarithmic moment estimator is only determined by data samples and it is computationally efficient owing to the explicit closed form. Table 2 shows the Monte Carlo simulation results based on the logarithmic moment method and the experimental conditions are the same as Table 1. Compared to the ratio estimator, logarithmic moment estimator leads to better performance in the same sample size.

Table 2 Performance of logarithmic moment estimator (true $\gamma = 1$)

<table>
<thead>
<tr>
<th>Number of Samples</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\alpha}$</td>
<td>$\hat{\gamma}$</td>
<td>$\hat{\gamma}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.5006 (0.0160)</td>
<td>1.0022 (0.0314)</td>
<td>1.5035 (0.0533)</td>
</tr>
<tr>
<td>2000</td>
<td>0.5001 (0.0111)</td>
<td>1.0008 (0.0243)</td>
<td>1.4985 (0.0350)</td>
</tr>
<tr>
<td>5000</td>
<td>0.4994 (0.0076)</td>
<td>1.0005 (0.0148)</td>
<td>1.5025 (0.0220)</td>
</tr>
</tbody>
</table>

3.4 Iterative logarithmic moment estimator

For the logarithmic moment estimator, all samples must be processed at one time. However, a reasonable choice is to update the estimated parameter values iteratively in order to achieve memory efficiency [7]. Let total samples be divided into $B$ non-overlapping blocks and each block contains $M$ samples. Denoting $\hat{\alpha}(k)$ and $\hat{\gamma}(k)$ as the estimated parameter values derived from the first $k$ ($1 \leq k \leq B$) blocks, we hope that $\hat{\alpha}(k)$ and $\hat{\gamma}(k)$ can be obtained from the previous estimated values $\hat{\alpha}(k-1)$ and $\hat{\gamma}(k-1)$. This is the main idea of the iterative logarithmic moment estimator. Donating $E_{1:k}$ as the sample mean calculated from the first $k$ blocks, after some manipulation, we can obtain

$$E_{1:k} = \frac{k-1}{k} E_{1:k-1} + \frac{1}{k} E_k. \quad (16)$$

Here, $E_k$ denotes the sample mean calculated from the $k$th block. Similarly, denoting $V_{1:k}$ as the sample variance calculated from the first $k$ blocks, we can write

$$V_{1:k} = \left(\frac{k-1}{kM-1} V_{1:k-1} + \frac{M-1}{kM-1} V_k + \left(\frac{k-1}{k} M \right) (E_{1:k-1} - E_k)^2 \right), \quad (17)$$

where $V_k$ denotes the sample variance calculated from the $k$th block. Substituting (14) and (15) into (16) and (17), we can obtain the iterative logarithmic moment estimator by

$$\frac{2 \log \hat{\gamma}(k)}{\hat{\alpha}(k)} = \frac{k-1}{k} \left[ 2C_\gamma \left(\frac{1}{\hat{\alpha}(k-1)}\right) + \frac{2 \log \hat{\gamma}(k-1)}{\hat{\alpha}(k-1)} + 2 \log 2 \right] + \frac{E_k}{k} - 2 \log 2 + 2C_\gamma \left(1 - \frac{1}{\hat{\alpha}(k)}\right)$$

and

$$\hat{\alpha}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(k-1) + \frac{2 \log \hat{\gamma}(k-1)}{\hat{\alpha}(k-1)} + 2 \log 2}$$

$$+ \frac{E_k}{k} - 2 \log 2 + 2C_\gamma \left(1 - \frac{1}{\hat{\alpha}(k)}\right)$$

(18)
\[
\frac{2\pi^2}{3\hat{\alpha}^2(k)} = \frac{(k-1)M-1}{kM-1} \frac{2\pi^2}{3\hat{\alpha}^2(k-1)} + \frac{M-1}{kM-1} V_k + \frac{(k-1)M}{k(kM-1)} \left[ 2C \left[ \frac{1}{\hat{\alpha}(k-1)} - 1 \right] + 2 \log \hat{\gamma}(k-1) + 2 \log 2 - E_k \right]^2 \]

Obviously, \( \hat{\alpha}(k) \) can be obtained from (19) then \( \hat{\gamma}(k) \) can be obtained from (18). Table 3 shows the Monte Carlo simulation results based on the iterative logarithmic moment estimator with 100 independent realizations. We can see that the number of data block is a key factor determining the performance of this iterative estimator, which is illustrated in Fig. 2. Obviously, better performance is achieved with bigger block size. The performance of this iterative estimator is similar to that of the logarithmic moment estimator in same sample size (e.g., 5000 samples), but the data required in each iterative computation is much fewer. This demonstrates the high efficiency of the iterative logarithmic moment estimator.

Table 3 Performance of iterative logarithmic moment estimator (true \( \alpha = 0.5, \gamma = 1 \))

<table>
<thead>
<tr>
<th>Samples of each Block</th>
<th>B = 50</th>
<th>B = 100</th>
<th>B = 200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\alpha} )</td>
<td>0.5004 (0.0093)</td>
<td>0.4997 (0.0073)</td>
<td>0.5004 (0.0055)</td>
</tr>
<tr>
<td>( \hat{\gamma} )</td>
<td>1.0014 (0.0204)</td>
<td>0.9990 (0.0156)</td>
<td>0.9990 (0.0113)</td>
</tr>
<tr>
<td>( \hat{\alpha} )</td>
<td>0.5007 (0.0070)</td>
<td>0.4999 (0.0049)</td>
<td>0.5004 (0.0037)</td>
</tr>
<tr>
<td>( \hat{\gamma} )</td>
<td>1.0007 (0.0157)</td>
<td>1.0010 (0.0108)</td>
<td>0.9998 (0.0073)</td>
</tr>
<tr>
<td>( \hat{\alpha} )</td>
<td>0.5007 (0.0048)</td>
<td>0.5004 (0.0037)</td>
<td>0.5003 (0.0026)</td>
</tr>
<tr>
<td>( \hat{\gamma} )</td>
<td>1.0003 (0.0108)</td>
<td>1.0004 (0.0065)</td>
<td>0.9997 (0.0051)</td>
</tr>
</tbody>
</table>

Fig. 2 Performance of iterative logarithmic moment estimator with respect to the number of data blocks: (a) Average of \( \hat{\alpha} \) compared with true value \( \alpha = 0.5 \); (b) Standard deviation of \( \hat{\alpha} \); (c) Average of \( \hat{\gamma} \) compared with true value \( \gamma = 1 \); (d) Standard deviation of \( \hat{\gamma} \).

4 Conclusion

Heavy-tailed exponential distribution, which is a useful tool for the modeling of SAR intensity images, is a necessary extension of the heavy-tailed Rayleigh distribution. First, we introduce some basic properties of the heavy-tailed exponential distribution such as the series form of density function, the heavy-tailed property and the non-existence of finite variance in this paper. Second, we propose three estimators based on negative-order moments and compare their performances. The ratio estimator greatly relies on the choice of order, and its performance is degraded seriously when the order is chosen inappropriately. The logarithmic moment estimator with explicit closed form is only determined by samples, and the iterative logarithmic moment estimator achieves better performance only using fewer samples in each step computation. Monte Carlo simulation results demonstrate that the iterative logarithmic moment estimator is high efficient for the parameter estimation of heavy-tailed exponential distribution.
References


