A Cooperative Game-Theoretic Measurement Allocation Algorithm for Localization in Unattended Ground Sensor Networks

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Abstract—This paper proposes a cooperative game-theoretic approach for efficient measurement allocation in unattended ground sensor networks when they are engaged in localizing a target. The game-theoretic approach evolves around the idea that localization is achieved as a result of collaboration among the nodes. Hence this process can be modeled as a cooperative game and the solution concept of the Shapley value can be exploited to determine the value of each node for localization. Furthermore, it is proved that by iteratively allocating measurements according to the Shapley value, stochastic observability - a measure of the predicted level of accuracy in localization - desirably improves.

Keywords: Unattended ground sensor networks, Localization and tracking, Sensor management, Dynamic cooperative (coalitional) games.

I. INTRODUCTION

Node management in unattended ground sensor (UGS) networks has recently received a sizeable attention due to the emergence of these networks as a robust solution for surveillance and reconnaissance in remote and inaccessible places [1]. Node management is an essential issue in these networks because the small size of the nodes severely limits the amount of energy that can be held on board.

In this paper, node management is studied in the context of measurement allocation to the nodes for localization. We consider a scenario in which localization is done by measuring the Direction of Arrival (DOA) of the signal emitted from the target. Extracting the target position from the DOA measurements is a form of triangulation and therefore highly nonlinear. The level of accuracy varies with the selection of the nodes and number of measurements each node takes.

Our main goal is to find efficient measurement allocations when the total number of measurements and target prior density are given. Since the level of accuracy cannot be known before measurements are taken, our objective function is the predicted level of accuracy. This measure is quantified by the determinant of the Posterior-Cramer Rao Lower Bound (P-CRLB) [2] and we refer to it as stochastic observability [3] or observability for short.

Main Results: We use a dynamic cooperative game-theoretic framework for allocating measurements to the nodes. Localization is essentially achieved by collaboration among the nodes. In this collaboration, the degree of contribution of each node can differ. It is therefore reasonable to allocate more measurements to the nodes which contribute more. This framework mimics a capitalist society in which the earnings are mostly reinvested where more profit is expected. It also replicates what is known as natural selection in population genetics [4]. To quantify the contribution of each node, we use the solution concept of the Shapley value (see Section III-A). To this end, we assume that the nodes are involved in a cooperative game with observability defined as the collective reward and we compute the Shapley value. We show by repeatedly allocating the measurements proportional to the Shapley value, observability increases. In general, the complexity of computing the Shapley value increases exponentially with the number of players. However, we show that the target localization game belongs to the class of weighted-graph games (see Section III-B) for which the Shapley value can be computed in polynomial-time.

Literature Review: The problem of node management for localization has been recently studied in two papers [5] and [6]. In these works, the goal is to find the best subset of the nodes which maximizes the reciprocal of the trace of the P-CRLB for a given set cardinality. To this end, problem-specific heuristic search methods are proposed. An assumption made in [5] and [6] is that each node only takes one measurement at each decision instant. We allow more measurements to be taken from the same node at each decision instant. Examples can be constructed to show repeated measurements from the same nodes provide better results than limiting to only one measurement from each node. Information-theoretic approach to node management of UGS networks for localization and tracking has also been studied in [7]–[10] where again heuristic search methods have been used to activate the nodes. For instance, in [7], only a single node is activated at a time. It can be shown that the information-theoretic approach is closely related to the lower-bound approach used in this paper [11]. In other works such as [12], [13], node management is studied qualitatively for a small number of the nodes. The problem of path planning for an observer in bearings-only tracking [14], [15] shares the same objective function with the problem described here but the decision variables and constraints are evidently very different.

Organization: The rest of this paper is organized as follows. In Section II, the problem of measurement allocation is
formulated by defining the measurement model and objective function. Since we take a cooperative game-theoretic approach for solving this problem, in Section III, we provide the relevant materials from cooperative game theory. Section IV incorporates the ideas from the previous two sections to solve the measurement allocation problem. In Section V, the numerical results are provided. Finally, some concluding remarks are made in Section VI.

II. FORMULATION OF THE MEASUREMENT ALLOCATION PROBLEM

The aim of this section is to formulate the measurement allocation problem. We first provide our modeling assumptions and then derive the formulation.

We consider a network $\mathcal{N} = \{1, 2, \cdots, N\}$ with the location of its nodes denoted by $[x_i, y_i]^T$ for $1 \leq i \leq N$. The network is to estimate the position of a target located at $X_t = [x_t, y_t]^T$ by using a total of $M$ noisy DOA measurements. We are to determine $m = [m_1, \cdots, m_N]$ the fraction of measurements that each node must take. Note that $m_i = M_i/M$ where $M_i$ is the actual number of measurements that node $i$ takes.

The true DOA for node $i$ can be found via

$$\theta_i = \text{atan2}(y_i - y_t, x_i - x_t)$$

(1)

where $\text{atan2}(y, x)$ is the arc tangent of $y/x$ in the interval $[-\pi, \pi]$ and the sign of $\text{atan2}$ is determined by the sign of $y$. The DOA measurement error is modeled by additive noise, i.e.

$$z_i = \theta_i + v_i$$

(2)

where we assume $v_i$ has a Gaussian distribution $N(0, \sigma_i^2)$ and it is an independent random variable for each node.

Let $\hat{X}_i(\mathcal{N}, m)$ denote the estimate of $X_t$ obtained from the allocations given by $m$. The P-CRLB theorem [2] states that the covariance of this estimate is lower bounded by the inverse of a matrix denoted by $J_n(\mathcal{N}, m)$ where

$$J_n(\mathcal{N}, m) = \mathbf{Q} + \mathbb{E}\{J(\mathcal{N}, m)\}.$$  (3)

Here $\mathbb{E}$ denotes the expectation operator and it is taken with respect to the prior density of the target $p(X_t)$. $\mathbf{Q}$ is given by

$$\mathbf{Q} = \mathbb{E}\{[\nabla_X\ln(p(X_t))][\nabla_X\ln(p(X_t))]^T\}$$

(4)

and $J(\mathcal{N}, m)$, the inverse of the CRLB, is given by

$$J(\mathcal{N}, m) =$$

$$\mathbb{E}\{[\nabla_X\ln(p(Z(\mathcal{N}, m)|X_t))][\nabla_X\ln(p(Z(\mathcal{N}, m)|X_t))]^T\}$$

(5)

where $Z(\mathcal{N}, m) = \{z_{i,j}\}_{i=1}^N$ is the set of the measurements that are taken from the nodes in $\mathcal{N}$, $p(Z(\mathcal{N}, m)|X_t)$ is the probability density of $Z(\mathcal{N}, m)$ conditioned on $X_t$, and $\nabla_X$ is the gradient operator with respect to $X_t$. Expectations in (4) and (5) are taken with respect to $p(X_t)$ and $p(Z(\mathcal{N}, m)|X_t)$ respectively. The conditional probability density in (5) is proportional to

$$p(Z(\mathcal{N}, m)|X_t) = \prod_{i=1}^N \left(\frac{1}{\sigma_i \sqrt{2\pi}}\right)^{M_i} \exp\left(-\frac{(z_i - \theta_i)^2}{2\sigma_i^2}\right).$$

(6)

Consequently, (5) can be written as

$$J_n(\mathcal{N}, m) = \frac{1}{\sigma_i^2} \left[\begin{array}{cc} \sin^2(\theta_i) - \sin(\theta_i)\cos(\theta_i) & \cos^2(\theta_i) - \sin(\theta_i)\cos(\theta_i) \\
\cos(\theta_i)\cos(\theta_i) - \sin^2(\theta_i) & \sin(\theta_i)\cos(\theta_i) - \cos^2(\theta_i) \end{array}\right]$$

(7)

where $r_i = \sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}$ is the relative range of the $i$-th node from the target.

The P-CRLB is a matrix. To summarize the characteristics of the P-CRLB in a real-valued function, we define observability as the determinant of the inverse of the P-CRLB, i.e.

$$\det(J_n(\mathcal{N}, m))$$

(8)

and

$$f_i = ME\{q_{11}\frac{\cos^2(\theta_i)}{\sigma_i^2}\} + q_{22}\frac{\sin^2(\theta_i)}{\sigma_i^2} + 2q_{12}\frac{\cos(\theta_i)\sin(\theta_i)}{\sigma_i^2}\}.$$ (9)

After disregarding the constant term $\det(Q)$ in (8), the measurement allocation problem can be stated as solving

$$\begin{align*}
\text{maximize} & \quad \frac{1}{2} m^T G m + F^T m \\
\text{subject to} & \quad \sum_{i=1}^N m_i = 1 \\
& \quad m_i \geq 0
\end{align*}$$

(10)

Since the constant term $\det(Q)$ is irrelevant in our measurement allocation problem, we interchangeably refer to the objective function in (11) as observability too. This objective function can be also written in purely quadratic terms by noting that

$$\begin{align*}
\frac{1}{2} m^T G m + F^T m &= \frac{1}{2} m^T (G + F1^T + 1F^T) m \\
&= \frac{1}{2} m^T \mathbf{W} m
\end{align*}$$

(11)

where we have used the fact that $\sum_{j=1}^N m_j = 1^T m = 1$. Now the objective function has the form $\mathbf{m}^T \mathbf{W} \mathbf{m}$ where

$$w_{ij} = g_{ij} + f_i + f_j.$$ (12)

It can be seen that the measurement allocation problem is a non-convex quadratic optimization problem because $G$ can be a positive definite, negative definite or indefinite matrix depending on the node positions and the prior density of the target.

III. TOOLS FROM COOPERATIVE GAME THEORY

To introduce the cooperative game-theoretic approach for measurement allocation, we review some relevant concepts from cooperative game theory in this section. Of particular interest is the solution concept of the Shapley value which is an indicator of the value of the players in a convex game. While a very pivotal solution concept in cooperative game theory,
the Shapley value is not very practical in many cases because
the computational complexity of the Shapley value increases
exponentially with the number of players. However, for a
class of cooperative games known as weighted-graph games to
which the target localization game belongs, the characteristic
function has symmetries and independencies that gives rise
to a more compact representation of the game and a more
efficient method for computing the Shapley value. We therefore
introduce weighted-graph games and compute their Shapley
value in this section. Throughout this section, we only study
static cooperative games and therefore simply refer to them as
cooperative games.

A. Solution Concepts of Convex Games:

A cooperative game is defined by a pair \((N, v)\), where
\(N = \{1, 2, \ldots, N\} \) is the set of the players and 
\(v : 2^N \to \mathbb{R} \) is the characteristic function which assigns a real value
to each subset (coalition) of the players. This value is the
collective reward of the coalition. The value of the empty set
is assumed to be zero, i.e. \(v(\emptyset) = 0\). The solution concepts
of cooperative games determine both the relative power of
different coalitions, as well as the strength of different players
within each coalition based on the characteristic function.

Most interesting cooperative games are superadditive. In
these games, we have
\[ v(S_1) + v(S_2) \leq v(S_1 \cup S_2) \] (14)
for any two disjoint subsets \(S_1\) and \(S_2\) in \(N\). This means that
the value of the entire set of players, i.e. the grand coalition,
is no less than the value of any other coalition and therefore
the players are inclined to form this coalition.

Two main solution concepts of superadditive games are the
core and the Shapley value. They prescribe how the collective
reward of the grand coalition should be allocated among the
players. The core is the set of all reward vectors which satisfy
two properties of acceptability and feasibility. The reward
vector \([\kappa_1, \kappa_2, \ldots, \kappa_N]^T\) is acceptable if \(\sum_{i \in S} \kappa_i \geq v(S)\) and
it is feasible if \(\sum_{i=1}^{N} \kappa_i = v(N)\). The core conveys the notion
of stability in the sense that there is no smaller coalition with
both the desire and the power to change the outcome of the
game.

As opposed to the core which is a set of reward vectors, the
Shapley value is a unique vector. The Shapley value embraces
the notion of fairness in the sense that if two players exactly
have the same marginal contributions to every coalition, their
Shapley values are equal. The marginal contribution of the
player \(i\) to the coalition \(S\) is defined as 
\(v(S \cup \{i\}) - v(S)\). The Shapley value of player \(i\) is the average of its marginal
contributions given by
\[ \rho_i = \sum_{S \subseteq N} \frac{S!(N - S - 1)!}{N!} (v(S \cup \{i\}) - v(S)) \] (15)
where \(S\) denotes the number of players in \(S\). It can be shown
that the sum of the Shapley values of all players adds up to
the value of the grand coalition, i.e.
\[ v(N) = \sum_{i=1}^{N} \rho_i. \] (16)

The core, in general, can be empty. However for a special
class of superadditive games, known as convex (or supermod-
ular), the core is always non-empty [16]. The characteristic
function of a convex game satisfies
\[ v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(S_1 \cap S_2) \] (17)
for any two subsets \(S_1\) and \(S_2\) in \(N\). It can also be shown
that the Shapley value of a convex game is in the core and in
fact is the centroid of the core [16]. In this regard, the Shapley
value of a convex game can be considered as the best indicator
of the value of the players.

B. Weighted-Graph Games:

A weighted-graph game is associated with an undirected
graph \((V, C)\) where \(V\) is the set of vertices and \(C\) is the matrix
of non-negative edge weights, and defined as a cooperative
game with
\[ N = V \]
\[ v(S) = \frac{1}{2} \sum_{i,j \in S} c_{ij} \quad \text{for} \quad S \subseteq N. \] (18)
For a cooperative game represented as a graph, the vertices
are the players, and the value of a coalition is obtained by
summing the weights of the edges that connect a pair of vertices
in the coalition with self-loop edges only considered with
half of their weights. A weighted-graph game can therefore
be represented compactly by \(\frac{N(N-1)}{2} + N\) weights, as opposed
to \(2^N\) numbers which are required normally to represent a
cooperative game. It is also not difficult to prove that a
weighted-graph game is convex. The following proposition
shows the computation of the Shapley value for weighted-
graph games can be done substantially faster than what is
generally possible for cooperative games when the number of
players is large.

**Proposition 1** The Shapley value of player \(i\) of a weighted-
graph game \((V, C)\) is given by
\[ \rho_i = \frac{1}{2} \sum_{j=1}^{N} c_{ij}. \] (19)

**Proof** See Appendix I.

IV. MAIN ALGORITHM

In this section, we first give the definition of the dy-
namic cooperative target localization game and investigates
its properties. As the main property of this game, we prove
that observability monotonically increases as the outcome of
this game. We then provide a comprehensive algorithm for
measurement allocation by integrating the target localization
game with the particle filter.
A dynamic cooperative game, in general, is a sequence of characteristic functions, i.e. static cooperative games \( \{v^k\} \), generated by an allocation rule
\[
h : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2^N}, \quad v^{k+1} = h(v^k),
\]
(20)
The allocation rule states how the collective reward is reinvested in the game. For instance, an allocation rule which mimics a capitalist society allocates most of the reward to those who contributed the most in generating that reward. This is usually done with the expectation that the future collective reward increases even further as a result. Since we are interested in maximizing observability, we construct the target localization game with a capitalistic-minded allocation rule.

**Definition 1** The dynamic cooperative target localization game is a sequence of weighted graph games with the elements of the weight matrices at stage \( k \) given by
\[
c^k_{ij} = m^k_{ij}w_{ij}m^k_{ij},
\]
where \( w_{ij} \) is derived from (13) and the allocation rule given by
\[
m^{k+1} = \frac{m^k}{\sum_{j=1}^N m^k_j}.
\]
(22)

Using (19), we can compute the Shapley value at each stage of the target localization game via
\[
\rho^k_i = \frac{1}{2} \sum_{j=1}^N m^k_{ij}w_{ij}m^k_j.
\]
(23)

Hence the measurement allocation rule in (22) can be written directly in terms of the past allocations
\[
m^{k+1} = \frac{m^k}{\sum_{j=1}^N \sum_{q=1}^P w_{pq}m^k_{ij}m^k_j}.
\]
(24)

We denote the value of the characteristic function for coalition \( S \) and allocation \( m^k \) in stage \( k \) by \( v(S, m^k) \), i.e.
\[
v(S, m^k) = \frac{1}{2} \sum_{i,j \in S} m^k_{ij}w_{ij}m^k_j.
\]
(25)

A quick review of the definition of the target localization game reveals that \( v(N, m) \) is the objective function in (11) and the allocation rule (22) automatically satisfies the constraints of (11). Also note that the allocation rule is based on the Shapley value of the nodes. Hence it reallocates the measurements among the nodes proportional to their value for localization. The following proposition formally proves that the allocation rule in (22) increases observability.

**Proposition 2** Observability, i.e. \( v(N, m^k) \) as defined in (25), increases monotonically after each stage of the target localization game.

**Proof** See Appendix I.

We can now present the comprehensive cooperative game-theoretic measurement allocation algorithm for localization in a sensor network by integrating the target localization game with a sequential Bayesian estimator. In general, any Bayesian estimator can be used with the target localization game, however, we have particularly chosen the particle filter because of its reported superior performance compared to other Bayesian estimators in bearings-only tracking [17]. In the algorithm given below, \( \{x^p_i, y^p_i \}^{L}_{i=1} \) denotes the particles and their weights at decision instant \( t \). The relative range and DOA of node \( i \) from particle \( j \) at decision instant \( t \) are denoted by \( r_{ij}^{t, c} \) and \( \theta_{ij}^{t, c} \) respectively.

**Measurement Allocation Algorithm for UGS Networks**

*Given: Prior density of the target represented by \( \{x^1_i, y^1_i \}^{L}_{i=1} \) and node positions \( \{x^t_i, y^t_i \}^{N}_{i=1} \).*

**Initialization:** \( t = 1 \).

1. Compute \( Q \approx \text{cov}(x^t_i) \) \(^{-1} \).
   
   \[ Q \approx M \sum_{c=1}^L \frac{\text{cov}(x^t_i)}{\sigma_{x^t_i}^2} \sigma_{x^t_i} \sigma_{x^t_i} \]
   
   cf. (4)

2. Compute \( r_{ij}^{t, c} = \sqrt{(x^t_i - x^t_j)^2 + (y^t_i - y^t_j)^2} \) and \( \theta_{ij}^{t, c} = \text{atan2}(y^t_i - y^t_j, x^t_i - x^t_j) \).

3. Compute
\[
g_{ij} = M^2 \text{E}[\frac{\sin^2(\theta_{ij} - \theta_{ij}^t)}{\sigma_{\theta_{ij}}^2}] = M^2 \sum_{d=1}^L \sum_{c=1}^L \frac{\sin^2(\theta_{ij}^{t, c} - \theta_{ij}^{t, d})}{\sigma_{\theta_{ij}^{t, c}}^2}\sigma_{\theta_{ij}^{t, d}}\sigma_{\theta_{ij}^{t, d}}.
\]

4. Compute
\[
f_i = M \text{E}[q_{11} \frac{\cos^2(\theta_{ij})}{\sigma_{\theta_{ij}}^2} + q_{22} \frac{\sin^2(\theta_{ij})}{\sigma_{\theta_{ij}}^2} + 2q_{12} \frac{\cos(\theta_{ij})\sin(\theta_{ij})}{\sigma_{\theta_{ij}}^2}]\]
\[
\approx M \sum_{c=1}^L \frac{\cos^2(\theta_{ij}^{t, c})}{\sigma_{\theta_{ij}^{t, c}}^2}\frac{\sin^2(\theta_{ij}^{t, c})}{\sigma_{\theta_{ij}^{t, c}}^2} + 2q_{12} \frac{\cos(\theta_{ij}^{t, c})\sin(\theta_{ij}^{t, c})}{\sigma_{\theta_{ij}^{t, c}}^2}.
\]

5. Let \( w_{ij} = g_{ij} + f_i + f_j \).

6. Let \( m^t = \frac{1}{M} \text{E}\) and \( k = 1 \).

6.1. Compute the Shapley value \( \rho^k_i = \frac{1}{2} \sum_{j=1}^N m^k_{ij}w_{ij}m^k_j \).

6.2. Compute the new allocation \( m^k_{ij} = \frac{\rho^k_i}{w_{ij}m^k_j} \).

6.3. If the convergence criterion is not satisfied, let \( k = k + 1 \) and go to Step 6.1. Otherwise, let \( m^* = m^k \) and go to Step 7.
7. Collect a set of new measurements, \( Z'(N, \mathbf{m}^*) \), from the nodes according to \( \mathbf{m}^* \).
8. Run the particle filter \( \{ \chi_{t+1,i}, \eta_{t+1,i} \}_{i=1}^L = \text{Particle Filter}(\{ \chi_{t,i}, \eta_{t,i} \}_{i=1}^L, \mathbf{Z}'(N, \mathbf{m}^*)) \).
9. \( t = t + 1 \) and go to step 1.

V. NUMERICAL EXAMPLES

In this section, we analyze the behavior of the measurement allocation algorithm in two case studies. In the first case study, we analyze its behavior in a small network with 5 nodes. The small size of the network gives insight on how the configuration of the network and target plays a role in measurement allocation. In the second case study, we investigate the influences of the prior density of the target in measurement allocation. For this purpose, we study the localization of a target located at the center of a circular network. The symmetry in this target and network configuration removes the initial effects of the node positions in measurement allocation. Throughout this section, a standard deviation of 10 degrees was assumed for the nodes, i.e. \( \sigma_i = 10 \).

A. Effects of Network and Target Configuration on Measurement Allocation:

For this case study, we consider the configuration shown in Fig. 1 which consists of a network with 5 nodes and a target located at \([0, 0]^T\). We ignore the effects of the prior density and only aim at finding the allocations for the given position of the target. Formally, this means we optimize the determinant of \( \mathbf{J}(N, \mathbf{m}) \) in (5) instead of the determinant of \( \mathbf{J}_n(N, \mathbf{m}) \) in (3).

Following the steps similar to the ones shown in Section II, one can show that in this case \( f_i = 0 \) and \( g_{ij} = \frac{\sin^2(\theta_i - \theta_j)}{\sigma_i^2 + \sigma_j^2 + \sigma_{ij}^2} \).

As a result, \( \mathbf{W} \), see (13), is given by

\[
10^{-3} \times \begin{bmatrix}
    0 & 1.97 & 6.38 & 3.25 & 2.49 \\
    1.97 & 0 & 3.33 & 1.70 & 1.30 \\
    6.38 & 3.33 & 0 & 0 & 0 \\
    3.25 & 1.70 & 0 & 0 & 0 \\
    2.49 & 1.30 & 0 & 0 & 0
\end{bmatrix}
\]

for this configuration. Note that the zero elements in \( \mathbf{W} \) correspond to the nodes that have an identical line of sight with respect to the target. To illustrate how the Shapley value is an indicator of the value of the nodes for localization, we first argue qualitatively on the subject and then show that the Shapley values confirm our observations. To have a fair ground for comparison, we assume all nodes take only one measurement. We note that nodes 3, 4 and 5 all have the same line of sight with the respect to the target. Therefore the information that these nodes provide for localization must be somewhat redundant. Additionally, nodes 4 and 5 are farther from the target than node 3. Hence the contribution of these nodes should be even less significant than that of node 3. These observations, in fact, can be confirmed by looking at the Shapley values which are given in Table I and computed using (23). According to the table, nodes 1, 2, and 3 are more valuable for localization than node 4 and 5.

Further application of (24) reveals that allocations eventually converge to \([\frac{1}{3}, 0, \frac{1}{3}, 0, 0]^T\). To visualize these results, we can alter the network shown in Fig. 1 by removing nodes 4 and 5 and only considering nodes 1, 2, and 3. The space of all possible allocations for a 3-node network can be represented by an equilateral triangle. The perpendicular distance of a point from a side of the triangle determines the allocation given to the vertex facing that side. Fig. 2 shows the orbit of \([\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]^T\) for this network. As expected, the orbit has eventually converged to \([\frac{1}{3}, 0, 0]^T\). This outcome can be also justified by noting that while nodes 1, 2, and 3 are relatively at the same distance from the target, the lines of sight of node 1 and 3 are perpendicular to each other. Hence better observability is provided by these two nodes.

![Fig. 1](image1.png)

Fig. 1. The target and network configuration for Case Study A. The target and nodes are marked by + and □ respectively. Nodes 4 and 5 provide redundant information about the target and hence their Shapley value is lower compared to other nodes.

**TABLE I**

<table>
<thead>
<tr>
<th>Node number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shapley value ((10^{-3} \times))</td>
<td>1.82</td>
<td>1.66</td>
<td>1.94</td>
<td>0.99</td>
<td>0.76</td>
</tr>
</tbody>
</table>

![Fig. 2](image2.png)

Fig. 2. The orbit of \([\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]^T\) where triangle coordinates are used to represent the space of allocations in a 3-node network. The orbit has eventually converged to \([\frac{1}{3}, 0, 0]^T\) which means in this case measurements must be equally taken from node 1 and 3.
B. Effects of Prior Density on Measurement Allocation:

We consider the circular network shown in Fig. 3 with a target at the center of it. This configuration may not be very realistic as the nodes are most often randomly dispersed in the region of the interest but it serves as an illustrative example to show how the prior density of the target position influences measurement allocation. At decision instant 1, the prior density was assumed to be a zero-mean Gaussian distribution with a very large covariance, i.e.

\[ p(X^t_1) \sim N(0, [100, 0; 0, 100]). \]  

(26)

We used this prior to calculate the allocations. Ideally because the prior density is symmetric, all nodes should be allocated with equal fraction of measurements. However, as \( G \) and \( F \) have to be computed numerically (see steps 3 and 4 of the measurement allocation algorithm), the allocations become unequal. This numerical error can be seen in Fig. 3(a) where the height of a black bar above a node indicates the relative size of its allocation. Based on these allocations, new measurements were obtained and the posterior target density was computed. The posterior density was approximated with a Gaussian distribution and used as the prior at the next decision instant. The ellipse in Fig. 3(a) represents the 1-\( \sigma \) confidence boundary of the posterior density. This boundary is defined as points \((x, y)^T\) which satisfy

\[ \begin{bmatrix} x - \hat{x}^t \ y - \hat{y}^t \end{bmatrix} \begin{bmatrix} \Sigma^t \end{bmatrix}^{-1} \begin{bmatrix} x - \hat{x}^t \ y - \hat{y}^t \end{bmatrix} = 1. \]  

(27)

Here \( [\hat{x}^t, \hat{y}^t]^T \) and \( \Sigma^t \) are the mean and covariance of the posterior density at decision instant \( t \).

Fig. 3(a) reveals that there is more uncertainty along the south-west, north-east direction. Therefore one would expect at the next decision instant, the nodes perpendicular to this direction are given larger allocations. This can be indeed confirmed by inspecting Fig. 3(b). Similar patterns are also seen in Figs. 3(c) and 3(d) for \( t = 3 \) and 4.

VI. CONCLUSIONS

The analytical and numerical results of this paper show that the Shapley value is an efficient indicator of the value of a node for localization in unattended ground sensor networks. For tracking applications, the predicted probability density of the target at each this decision instant can be used in (9) and (10) for the computation of expectations. In the absence of full knowledge about the target model, a Robbins-Monro stochastic approximation formulation [18] of (24) can be developed where allocations are recursively updated over time from the latest estimate of the target position without necessarily knowing or computing the expectations in (9) and (10).

APPENDIX I

PROOF OF PROPOSITIONS

A. Proof of Proposition 1

First, we consider the contribution of the edge \( c_{ii} \) to \( \rho_i \). For every subset \( S \) that does not contain \( i \), with the addition of \( i \), the edge \( c_{ii} \) contributes \( \frac{1}{2} \frac{N!}{s!(N-s)!} c_{ii} \) to \( \rho_i \). There are \( \frac{N-1}{S} \) subsets of size \( S \) that do not contain \( i \) and \( S \) can be any number between 0 and \( N-1 \). Hence the overall contribution of \( c_{ii} \) is

\[ \frac{1}{2} \sum_{s=0}^{N-1} \binom{N-1}{s} \frac{(N-s)!}{N!} c_{ii} = \frac{1}{2} c_{ii} \]

(28)

For the edge \( c_{ij} \), we consider the subset \( S \) which contains \( j \) but not \( i \). With the addition of \( i \) to \( S \), the edge \( c_{ij} \) contributes \( \frac{N!}{s!(N-s)!} c_{ij} \) to \( \rho_j \). There are \( \frac{N-2}{S-1} \) subsets of size \( S \) that contain \( j \) but not \( i \) and \( S \) can be any number between 1 and \( N-1 \). Hence the overall contribution of \( c_{ij} \) is

\[ \sum_{s=1}^{N-1} \binom{N-2}{s-1} \frac{(N-s)!}{N!} c_{ij} = \frac{1}{2} c_{ij}. \]  

(29)

B. Proof of Proposition 2

We have

\[ v(\mathcal{N}, m^{k+1}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} m_i^{k+1} m_j^{k+1} \]

\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \rho_i^k \rho_j^k \]

\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \frac{\rho_i^k \rho_j^k}{\sqrt{8v(\mathcal{N}, m^k)}} v(\mathcal{N}, m^k) \]

\[ = \frac{1}{8v(\mathcal{N}, m^k)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \sum_{p=1}^{N} w_{ip} m_i^k m_p^k \sum_{q=1}^{N} w_{jq} m_j^k m_q^k. \]

(30)

Therefore to prove that

\[ v(\mathcal{N}, m^{k+1}) \geq v(\mathcal{N}, m^k), \]

(31)

one must show

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \sum_{p=1}^{N} w_{ip} m_i^k m_p^k \sum_{q=1}^{N} w_{jq} m_j^k m_q^k \]

\[ \geq 8v(\mathcal{N}, m^k)^3 = \left( \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} m_i^k m_j^k \right)^3. \]

(32)

This inequality has been proved in [19]. It has also been shown that equality in (32) is obtained only at points where \( m_i^k = 0 \) or

\[ \sum_{j=1}^{N} w_{ij} m_j^k = \sum_{p=1}^{N} w_{pq} m_p^k m_q^k \]

for all \( 1 \leq i \leq N \). It can be easily confirmed that the points which satisfy these conditions are the equilibria of (24).
Fig. 3. The localization of a target located at the center of a circular network. The black bar above each node is proportional to the fraction of measurements allocated to that node. The ellipses represent the 1-σ confidence boundary, see (27).

REFERENCES


