Abstract—Track-to-Track Association (T2TA) is very important in distributed tracking systems. When the fusion center receives multiple tracks from a local tracker, T2TA needs to be performed before Track-to-Track Fusion (T2TF). The problem of T2TA, similar to measurement-to-track association, can be formulated and solved as an assignment problem, whose costs are based on the likelihood of assigning a local track to a central track, namely, the likelihood of the hypothesis $H_0$: the two tracks originated from the same target. Thus the basic hypothesis test of whether two tracks are for the same target is very important for T2TA. Compared to the measurement-to-track association, T2TA has two features: i) The local track is correlated with the central track that has the same origin with it, due to the common process noise of the target. ii) Multiple frames of data are available from both the tracks. Thus, two issues are of concern for the T2TA test, namely, the derivation of the exact test statistic needs to take into account the crosscorrelation between the tracks; and how should multiple frames of data from the tracks be utilized.

In this paper, the exact algorithm for calculating the test statistic for T2TA using multiple frames of data is derived. To keep the complexity of the algorithm under control, a limit is set for the number of the frames of data used in the test, which leads to a sliding window test for the problem of T2TA. By accounting for the crosscorrelations between tracks for the same target and the crosscorrelations across time of the terms entering into the test statistics, the proposed test for T2TA yields false rejections of $H_0$ that match the theoretical values. Then the sliding window test is compared with the single time test. It is shown that the intuitive belief “the longer the window, the more the power of the test” is not necessarily correct. Thus, caution should be used when using multiple frames of data from the tracks to improve the power of the test.

Keywords: Tracking, Track-to-track association, Kalman filtering, Hypothesis test.

I. INTRODUCTION

In a multisensor tracking system, the fusion center is meant to gather and process information from the sensor network. There are generally two approaches for this purpose. One is to send the local measurements directly to the fusion center. Another approach is to send local tracks to the fusion center where track-to-track fusion (T2TF) is to be performed. Studies on the problem of T2TF can be found in [7], [8], [12], [13]. Compared to transferring local measurements to the fusion center, the transfer of local tracks can be performed at a lower rate, which may lead to significant savings in communication. This advantage of T2TF allows it to be implemented in a distributed tracking system with low communication capacities [14]. Before T2TF is done at the fusion center, the problem of track-to-track association (T2TA) needs to be solved first. In general, the data association problem (measurement-to-track association, or T2TA) can be formulated as an assignment problem which seeks to find the associations that yields the highest overall likelihood. However, one basic problem is to evaluate the likelihood of a certain association, e.g., the likelihood of associating a local track to a central track in T2TA. This is directly related to the hypothesis test of whether two tracks originated from the same target. Unlike the measurement-to-track association, a particular challenge associated with this test for T2TA is to account for the crosscorrelation between the tracks of the same target due to the common process noises [3]. Studies on the problem of T2TA can be found in [4], [11], in which the tests for T2TA are made based on a single frame of data. In [10], it was claimed that the test based on the average of the single time tests within a certain time window has improved performance over the single time test. However, this conclusion was drawn ignoring the crosscorrelations between the single time statistics. In this paper, the exact test for T2TA based on multiple frames of data is presented. It is shown that a sliding window test for T2TA may be counterproductive and has, in some situations, less power than the single time test. This is due to the strong correlations between the individual frames of the data in such situations. Thus caution should be used when using multiple frames of data from the tracks to improve the power of the test.

The paper is organized as follows. Section II formulates the problem and gives the exact algorithm of calculating the test statistics. Section III specifies the hypothesis test for the problem of T2TA. In section IV, The performance of the exact sliding window test, the single time test and other two inexact tests which ignore the crosscorrelations between the single time tests are compared based on their false alarm rate and power. It is shown that the inexact tests lead to a false alarm rate that is significantly larger than the theoretical value. It is also observed that, with the same false alarm rate, the sliding
window test does not necessarily have more power than the single time test. Section V analyzes further the power of the sliding window test vs. the single time test. The results provide a guideline for using multiple frames of data from the tracks to improve the power of the test. Concluding remarks are provided in section VI.

II. THE EXACT ALGORITHM FOR THE SLIDING WINDOW TEST STATISTIC

Consider the simple T2TA problem of testing whether a central track and a local track originated from the same target (at a single time \( k \)). The information for the test consists of the central estimate \( \hat{x}_c(k|k) \) with covariance \( P_c(k|k) \), the local (sensor) estimate and covariance \( \hat{x}_s(k|k), P_s(k|k) \), as well as the crosscovariance \( P_{cs}(k|k) \) [1], [4]. Define

\[
\Delta(k) = \hat{x}_c(k|k) - \hat{x}_s(k|k)
\]

and

\[
P_\Delta(k) = P_c(k|k) + P_s(k|k) - P_{cs}(k|k)'
\]

To calculate \( P_{cs}(k|k) \), suppose the previous central and local tracks at time \( l \) are available at the fusion center. The errors of the tracks are

\[
\tilde{x}_c(l|l) = \hat{x}_c(l|l) - x(l)
\]

\[
\tilde{x}_s(l|l) = \hat{x}_s(l|l) - x(l)
\]

where \( x(l) \) is the true state of the target at \( l \). Let

\[
P_c(l|l) = \text{Cov}[\tilde{x}_c(l|l), \tilde{x}_c(l|l)]
\]

(5)

\[
P_s(l|l) = \text{Cov}[\tilde{x}_s(l|l), \tilde{x}_s(l|l)]
\]

(6)

\[
P_{cs}(l|l) = \text{Cov}[\tilde{x}_c(l|l), \tilde{x}_s(l|l)]
\]

(7)

From Eq. (8.4.2-2) in [1], one has

\[
\tilde{x}(l + 1|l + 1) = [I - K(l + 1)H(l + 1)]F(l)\tilde{x}(l|l) - [I - K(l + 1)H(l + 1)]v(l) + K(l + 1)w(l + 1)
\]

(8)

Let \( \tilde{x}_c(l|l) \) and \( \tilde{x}_s(l|l) \) denote the errors of the central track and local track \( s \) at time \( l \) before \( k \). Using (8) recursively for both the central and local tracker \( s \) from \( l \) to \( k \), the errors are propagated to \( k \) as

\[
\tilde{x}_c(k|k) = W^c_c(k, l)\tilde{x}_c(l|l) + \sum_{i=l+1}^{k} W^c_c(k, i-1)v(i-1) + \sum_{i=l+1}^{k} W^c_c(k, i)w_c(i)
\]

(9)

\[
\tilde{x}_s(k|k) = W^s_s(k, l)\tilde{x}_s(l|l) + \sum_{i=l+1}^{k} W^s_s(k, i-1)v(i-1) + \sum_{i=l+1}^{k} W^s_s(k, i)w_s(i)
\]

(10)

where \( v(i-1), i = l + 1 \ldots k \) are the process noise of the target; \( w_c(i) \) and \( w_s(i) \) are the noises of the central and local measurements. Eqs. (9) and (10) are the expressions of the errors of the tracks from Kalman filter as weighted sums of a previous error at a certain point, the process noises and the measurement noises afterwards. The weights, without indicating central or local track, are given by

\[
W_c(k, l) = \prod_{i=0}^{k-l-1} [I - K(k - i)H(k - i)]
\]

\[
\cdot F(k - i - 1)
\]

(11)

\[
W_s(k, i - 1) = \{- \sum_{j=0}^{k-i-1} [I - K(k - j)H(k - j)]
\]

\[
\cdot F(k - j - 1)\} I - K(i)H(i)
\]

(12)

\[
W_w(k, i) = \{ \sum_{j=0}^{k-i-1} [I - K(k - j)H(k - j)]
\]

\[
\cdot F(k - j - 1)\} K(i)
\]

(13)

where \( K(i), i = l + 1, \ldots, k \) are the Kalman filter gains, \( H(i) \) are the observation matrices and \( F(i - 1) \) are the state transition matrices. Assuming the measurement noises are white and uncorrelated with the process noises, and with \( P_c(l|l), P_s(l|l) \) and \( P_{cs}(l|l) \) available at the fusion center, the crosscovariance \( P_{cs}(k|k) \) is obtained as

\[
P_{cs}(k|k) = W^c_c(k, l)P_{cs}(l|l)W^s_s(k, l)'
\]

\[
+ \sum_{i=l+1}^{k} W^c_c(k, i-1)Q(i-1)W^s_s(k, i - 1)
\]

(14)

where \( Q(i-1) \) is the covariance of process noise; \( W^c_c(k, l) \) and \( W^s_s(k, i-1) \) are (11) and (12) evaluated from the central track, similarly \( W^c_c(k, l) \) and \( W^s_s(k, i-1) \) are the weights from local tracker \( s \). Notice that, in tracking systems with nonlinear measurement model, the filter gains and the measurement matrices of the local tracker \( s \), which are required for the calculation of \( W^c_c(k, l) \) and \( W^s_s(k, i-1) \), are not available at the fusion center. The transfer of these matrices, usually of high dimensions, are even more expensive than the transfer of the raw measurements. To avoid this problem, approximate evaluations of the crosscorrelation \( P_{cs}(k|k) \) are studied in literature, e.g., [6]. In the present paper, a method of approximating the local weights in (10) by reconstructing the local filter gains and local measurement matrices at the fusion center is proposed, which provides significant savings in communication and has no loss in actual performance [14] (see the Appendix for the details). With the reconstructed local information, \( W^c_c(k, l) \) and \( W^s_s(k, i-1) \) can thus be approximated at the fusion center, so \( P_{cs}(k|k) \) can be calculated from (14).

For T2TA based on a single time \( k \), the test statistic is

\[
T(k) = \Delta(k)'P_\Delta(k)^{-1}\Delta(k)
\]

(15)

which is a random variable with a \( \chi^2(n_x) \) distribution \( (n_x \) is the dimension of the state). However, for data association at subsequent times, the similar test statistics \( T(g), g > k \), are correlated with \( T(k) \), thus the sum of single time test statistics (15) within a time window does not have a \( \chi^2 \) distribution [4], [11].

\[1\text{This is contrary to the assertion in [10].}\]
The exact test based on multiple frames of data requires to calculate the crosscovariances among all the differences (1) and the inversion of the combined covariance matrix, which, under $H_0$, has an increasing dimensionality. In order to control the complexity of the test, the exact sliding window test based on the most recent $N$ frames of data is considered.

Without loss of generality, consider the $m$th association that occurs at time $t_m$. Define, using a subscript $N$ for the window length (unless it is one, i.e., single time),

$$\Delta_N(t_m) = [\Delta(t_m) \Delta(t_m-1) \ldots \Delta(t_m-N+1)]^T$$

(16)

The exact test statistic $T_N(t_m)$ for this sliding window of $N$ times is $\Delta$.

$$T_N(t_m) = \Delta_N(t_m) \text{Cov} [\Delta_N(t_m)]^{-1} \Delta_N(t_m)$$

(17)

which has a $\chi^2$ distribution with $Nn_x$ degrees of freedom.

Given $\text{Cov}[\Delta_N(t_m-1)]$ for the previous sliding window, to obtain $\text{Cov}[\Delta_N(t_m)]$, the new diagonal term

$$\text{Cov}[\Delta(t_m)] = P_{\Delta}(t_m)$$

(18)

is calculated as (2). For the new off-diagonal terms

$$\text{Cov}[\Delta(t_m), \Delta(t_i)] =$$

$$\text{Cov}[\tilde{x}_c(t_m,t_m), \tilde{x}_s(t_i,t_i)] + \text{Cov}[\tilde{x}_c(t_m,t_m), \tilde{x}_c(t_i,t_i)] - \text{Cov}[\tilde{x}_{sc}(t_m,t_i), \tilde{x}_{sc}(t_i,t_i)]$$

(19)

From (9) and (10), it follows that

$$\text{Cov}[\Delta(t_m), \Delta(t_i)] =$$

$$W_e^c(t_m,t_i)P_c(t_i|t_i) + W_s^c(t_m,t_i)P_s(t_i|t_i) - W_e^s(t_m,t_i)P_{sc}(t_i|t_i)$$

(20)

where $P_c(t_i|t_i)$, $P_s(t_i|t_i)$ and $P_{sc}(t_i|t_i)$ are available from the $i$th association.

III. THE HYPOTHESIS TEST FOR T2TA

Given a central track and a local track the hypothesis of whether the two tracks have the same origin is

$$H_0 : x_c(k) - x_s(k) = 0 \text{ at any time } k$$

$$H_1 : x_c(k) - x_s(k) \neq 0$$

where $x_c(k)$ and $x_s(k)$ are the true states of the targets corresponding to the central and the local tracks. The exact sliding window test is

Accept $H_0$, if the test statistic $T_N(t_m) \leq D_\alpha$

Accept $H_1$, if otherwise

$D_\alpha$ is the acceptance threshold that yields a false alarm rate of $\alpha$. Under $H_0$, the exact test statistic $T_N(t_m)$ (see (17)) has a $\chi^2$ distribution with $Nn_x$ degrees of freedom, whose cumulative distribution function has a closed form

$$P(\chi^2(Nn_x) < x) = \frac{\gamma(Nn_x/2, x/2)}{\Gamma(Nn_x/2)}$$

(21)

where $\Gamma$ denotes the Gamma function and $\gamma(n, x)$ is the lower incomplete Gamma function [15]. Thus $D_\alpha$ is chosen, such that

$$P(\chi^2(Nn_x) < D_\alpha) = \frac{\gamma(Nn_x/2, D_\alpha/2)}{\Gamma(Nn_x/2)} = 1 - \alpha$$

(22)

Software packages are available for the calculation of (21), e.g., [17], which is very useful in the design of the acceptance threshold $D_\alpha$.

The power of the test, however, has no theoretical solution. It depends on the actual difference of the true states of the targets which is unknown a priori and varies randomly due to different process noises of the targets. Also, under $H_1$ the distribution of the test statistic (17) is, in general, not $\chi^2$ and has no closed form, thus the theoretical value of the power of the test may not be obtained. In section V, a special case is studied, where, under $H_1$, two targets move in formation and have the same process noises. In such a scenario, the power of the test can be theoretically evaluated.

Besides the hypothesis test above, the likelihood of $H_0$, which corresponds to the likelihood of a individual assignment, is given by the probability density of the test statistic $T_N(t_m)$, which can be evaluated by the pdf (probability density function) of the $\chi^2$ random variable with $Nn_x$ degrees of freedom

$$p(x; Nn_x) = \left\{ \begin{array}{ll}
\frac{1}{2^{Nn_x/2} \Gamma(Nn_x/2)} x^{Nn_x/2 - 1} e^{-x/2} & \text{for } x > 0 \\
0 & \text{for } x \leq 0
\end{array} \right.$$ 

(23)

When multiple local tracks are to be assigned at the same time, the likelihoods of the individual assignments can be used to calculate the likelihood of the combined assignment. It is easy to see, the cost for the assignment problem which is based on this combined likelihood will be more effective, if the individual tests have more power given the same false alarm rate $\alpha$.

IV. SIMULATION RESULTS

The exact sliding window test (17) and the single time test (15) are compared next in a 1-D multisensor tracking scenario with two sensors and two targets. The target state is defined as $X = [x \ x \ \dot{x}]^T$. The motion of the targets is modeled as the DWNA model in [2], Section 6.3.2.

$$X(k+1) = FX(k) + Gv(k)$$

$$= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} X(k) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} v(k)$$

(24)

The covariance of the process noise is

$$Q = \begin{bmatrix} \frac{1}{2}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{bmatrix} \sigma_v^2$$

(26)
It is assumed the sensors obtain 1-D position measurements of the target with a sampling interval of $T = 1$ s. The measurement model is given by
\[
    z(k) = HX(k) + w(k)
    \]
\[
    = [1 \ 0]X(k) + w(k)
\]
\[
    V_{ar}[w(k)] = \sigma_w^2
\]
(27)

The standard deviation of the measurement noise is $\sigma_w = 30$ m. Target 1 starts at 5000 m with an initial velocity $v = 3$ m/s. Target 2 starts at 5030 m with the same initial velocity (the initial target separation is 30 m). Two targets have process noises with $\sigma_v^2 = 2 \cdot 10^{-2}$ m$^2$/s$^4$. The process noises are independent across the targets, leading to their eventual separation. Two sensors, designated as 1 and 2, obtain position measurements of targets 1 and 2, respectively, every 1 s and maintain separate tracks for the targets. The T2T associations are performed every 3 s. The sliding window test uses a window of $N = 5$ times. For comparison, two tests based on the sum of the single time tests are also performed. One is the cumulative sum over all previous single time tests; another is the sum of the single time tests within a sliding window of $N$ as the approach proposed in [10]. Both are not exact because the correlations in time are ignored.

Figure 1 shows the miss probability when the track of target 1 at sensor 1 is associated with the track of the same target at sensor 2. The theoretical miss probability is 2.5% (correct acceptance 97.5%). The “single time test” (based on single frame of data) and the “exact sliding window test” comply with their theoretical error probabilities. However, the miss probability of the other two tests based on the sum of the single time test statistics is significantly larger than the theoretical value. This is due to the fact that these tests ignore the correlation among the single time test statistics.

Figure 2 shows the probability of correct rejection (power of the test) for 0.025 miss probability when the track of target 1 at sensor 1 is associated with the track of target 2 at sensor 2. Surprisingly, the exact sliding window test has less power than the single time test. This counterintuitive phenomenon is further analyzed in the following section with a simple illustrative example.

V. THE EFFECT OF SLIDING WINDOW LENGTH ON THE T2T ASSOCIATION TEST

Consider a simple scalar state estimation problem with two sensors/estimators, and one target track at each. Estimator 1 has prior information $x_1(0) \sim N(\hat{x}_1(0), P_1)$, where $x_1(0)$ denotes the true state of the target corresponding to track 1 at time 0, $P_1$ is the variance of the estimate $\hat{x}_1(0)$. At time 1, this true state propagates to $x_1(1) = x_1(0) + v$ where $v \sim N(0,Q)$. A measurement is taken at time 1 as $z_1(1) = x_1(1) + w_1$, where $w_1 \sim N(0,R_1)$. Estimator 2 has prior information on the target corresponding to track 2, $x_2(0) \sim N(\hat{x}_2(0), P_2)$. At time 1, the state of this target evolves as $x_2(1) = x_2(0) + v$ and the measurement of sensor 2 is $z_2(1) = x_2(1) + w_2$, where $w_2 \sim N(0,R_2)$. It is assumed that the states of the two targets have the same process noise $v$, so the difference between their true states stays constant (they are moving in formation)$^3$. When the two targets are the same, $x_1(t) - x_2(t) = 0$, $t = 0, 1$, otherwise the target separation is $|x_1(t) - x_2(t)| = d > 0, t = 0, 1$. It is assumed that the errors in the prior information and the measurement noises are all independent and for the sake of simplicity $P_1 = P_2 = R_1 = R_2 = \sigma^2$.

For the T2T test based on the prior information at time 0, one has
\[
    \Delta(0) = \hat{x}_1(0) - \hat{x}_2(0)
\]
(29)

$^2$This small separation is only for the purpose of comparing the power of different tests for T2T association. It is assumed that these closely spaced targets are resolved by the sensors.

$^3$As it will be seen later, this assumption is necessary to obtain the actual theoretical performance of the exact test.
\[
\begin{align*}
\text{Var}[\Delta(0)] &= P_1 + P_2 = 2\sigma^2 \quad (30) \\
T(0) &= \text{Var}[\Delta(0)]^{-1} \Delta(0)^2 = \frac{1}{2\sigma^2} \Delta(0)^2 \quad (31)
\end{align*}
\]

under \(H_0\) (the two tracks are from the same target), \(E[\Delta(0)] = 0\); under \(H_1\) (the two tracks are from different targets), \(E[\Delta(0)] = d\).

At time 1, with the measurements \(z_1\) and \(z_2\), the updated estimates for the target (under \(H_0\)) or targets (under \(H_1\)) are

\[
\begin{align*}
\hat{x}_1(1) &= \frac{R_1}{P_1 + Q + R_1} \hat{x}_1(0) + \frac{P_1 + Q}{P_1 + Q + R_1} z_1 \\
\hat{x}_2(1) &= \frac{R_2}{P_2 + Q + R_2} \hat{x}_2(0) + \frac{P_2 + Q}{P_2 + Q + R_2} z_2
\end{align*}
\]

Thus

\[
\begin{align*}
\Delta(1) &= \hat{x}_1(1) - \hat{x}_2(1) \\
&= \frac{\sigma^2}{2\sigma^2 + Q}(\hat{x}_1(0) - \hat{x}_2(0)) \\
&\quad + \frac{\sigma^2 + Q}{2\sigma^2 + Q}(z_1 - z_2) \quad (32)
\end{align*}
\]

\[
\begin{align*}
\text{Var}[\Delta(1)] &= \frac{\sigma^4}{(2\sigma^2 + Q)^2} (P_1 + P_2) \\
&\quad + \frac{(\sigma^2 + Q)^2}{(2\sigma^2 + Q)^2} (R_1 + R_2) \\
&= \frac{2Q\sigma^2 + 4Q\sigma^4 + 4\sigma^6}{(2\sigma^2 + Q)^2} \quad (33)
\end{align*}
\]

\[
T(1) = \text{Var}[\Delta(1)]^{-1} \Delta(1)^2 \\
= \frac{(2\sigma^2 + Q)^2}{2Q\sigma^2 + 4Q\sigma^4 + 4\sigma^6} \Delta(1)^2 \quad (36)
\]

under \(H_0\), \(E[\Delta(1)] = 0\) and under \(H_1\), \(E[\Delta(1)] = d\).

For the sliding window test,

\[
\begin{align*}
\Delta_2(1) &= [\Delta(0) \Delta(1)]' \quad (37) \\
\text{Cov}[\Delta_2(1)] &= \begin{bmatrix}
2\sigma^2 & \frac{2\sigma^4}{2\sigma^2 + Q} \\
\frac{2\sigma^4}{2\sigma^2 + Q} & \frac{2Q\sigma^2 + 4Q\sigma^4 + 4\sigma^6}{(2\sigma^2 + Q)^2}
\end{bmatrix} \quad (38)
\end{align*}
\]

\[
T_2(1) = \Delta_2(1)' \{\text{Cov}[\Delta_2(1)]\}^{-1} \Delta_2(1) \quad (39)
\]

with the expectation taken conditioned on \(H_0\) (“same target”, i.e., \(d = 0\)) or \(H_1\) (\(d > 0\)). Specifically,

\[
T_N(t) \sim \chi^2(N, \lambda_N(t)), \quad t = 0, 1 \quad (41)
\]

Notice that (41) holds only when the covariance matrices of \(\Delta_N(t)\) are the same under both \(H_0\) and \(H_1\), which requires the targets to have the same process noise. This happens when the targets move in formation. However, in general, different targets do not necessarily have the same process noise. In such cases, the test statistic \(T_N(t)\) does not follow a non-central \(\chi^2\) distribution under \(H_1\) and the difference between the true states of the targets is nonstationary. Thus the power of the test can not be obtained theoretically.

The cumulative distribution function (cdf) of a \(\chi^2(N, \lambda)\) random variable is given by \([16]\)

\[
P \{ \chi^2(N, \lambda) \leq x \} = \sum_{j=0}^{\infty} e^{\lambda/2} \left( \frac{\lambda/2}{j!} \right) \frac{\Gamma(j + k/2)}{\Gamma(j + k/2)} \quad (42)
\]

Software packages are available for the calculation of (42), e.g., \([17]\).

The statistical properties of the above test statistics under \(H_0\) and \(H_1\) are shown in Table I. Notice that, in this example, the noncentrality parameter of the sliding window test \(T_2(1)\) doesn’t depend on the value of the process noise variance \(Q\). This is not true in general unless the tracks at different estimators have the same filter gains (which is the case in the example considered here). However, it is easy to see that the noncentrality parameter of the sliding window test is always greater than or equal to that of the single time test.

\[
\begin{array}{|c|c|c|}
\hline
\text{Test statistic} & \text{Degrees of freedom } N & \text{Noncentrality parameter } \lambda \\
\hline T(0) = T_1(0) & 1 & 0 \\
(\sigma^2 + 2Q \sigma^4 + 4\sigma^6) d^2 \\
T(1) = T_1(1) & 1 & 0 \\
(2Q \sigma^2 + 4Q \sigma^4 + 4\sigma^6) d^2 \\
T_2(1) & 2 & 0 \\
\hline
\end{array}
\]

Assuming \(\sigma^2 = 1\), Figure 3 compares the noncentrality parameters for the sliding window test (\(N=2\)) and the single time test at time 1. It can be seen that, if \(Q = 0\), then the noncentrality parameters of \(T_2(1)\) and \(T(1)\) are the same. However, \(T_2(1)\) has 1 more degree of freedom than \(T(1)\), thus the sliding window test \(T_2(1)\) requires a higher threshold for the same miss probability of \(H_0\) and, consequently, is less powerful than the single time test \(T(1)\). As the variance of the process noise \(Q\) increases, the noncentrality parameter of \(T_2(1)\) remains constant, and will be significantly larger than the noncentrality parameter of \(T(1)\), which decreases with \(Q\), see Figure 3. This compensates for the larger number of degrees of freedom of \(T_2(1)\) and makes \(T_2(1)\) eventually more powerful than \(T(1)\).
Figure 3. The noncentrality parameters (normalized by the separation squared) vs. process noise variance

Table II

<table>
<thead>
<tr>
<th>Test Stat</th>
<th>Q</th>
<th>N</th>
<th>λ</th>
<th>Threshold for rejection of H₀ (α = 0.025)</th>
<th>Theoretical power of the test</th>
<th>MC Miss Prob of H₀</th>
<th>MC Correct Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>T₁(1)</td>
<td>0.1</td>
<td>1</td>
<td>8.98</td>
<td>5.02</td>
<td>0.775</td>
<td>0.026</td>
<td>0.76</td>
</tr>
<tr>
<td>T₂(1)</td>
<td>2</td>
<td>9</td>
<td>7.38</td>
<td>0.678</td>
<td>0.030</td>
<td>0.69</td>
<td></td>
</tr>
<tr>
<td>T₁(1)</td>
<td>6</td>
<td>1</td>
<td>5.76</td>
<td>5.02</td>
<td>0.56</td>
<td>0.031</td>
<td>0.57</td>
</tr>
<tr>
<td>T₂(1)</td>
<td>2</td>
<td>9</td>
<td>7.38</td>
<td>0.678</td>
<td>0.029</td>
<td>0.66</td>
<td></td>
</tr>
</tbody>
</table>

Table II compares the power of the tests under different Q when d = 3 and σ² = 1. The “Threshold for rejection” and “Power of the test” are from (exact) theoretical calculations. The “Miss probability” and “Correct rejection” are from Monte Carlo simulations. The results show that the power of the test depends on (i) the number of degrees of freedom N (which determines the threshold) and (ii) the noncentrality parameter λ. In this example, the sliding window test T₂(1) has the same power over different process noise levels Q as a result of the constant noncentrality parameter. The single time test T₁(1), on the other hand, is less powerful than T₂(1) when Q = 6, but more powerful than T₂(1) when Q = 0.1. This shows that, for low level process noise, the single time test can be (for the same miss probability) more powerful than the sliding window test. The advantage of the window is negated in this case by the crosscorrelation in time which is higher for low process noise⁴. This suggests that, to enhance the power of the sliding window test, it is necessary to make sure that the multiple frames of data selected for the test are not strongly correlated. This can be accomplished by increasing the time difference between the selected frames.

To confirm this guideline, Figure 4 shows the power of the tests under a theoretical false alarm rate of α = 0.025 in the same simulation scenario as in section IV except that the tests for T2TA are done every 15 s as opposed to 3 s and the length of the sliding window is set to 4. It is shown that, in this case, the sliding window test has more power than the single time test. Also notice that, although the two inexact tests seem to have better power, as shown previously, their false alarm rates are much higher than the designed threshold α = 0.025.

VI. Conclusions

The hypothesis test for track-to-track association (T2TA) is studied in this paper. First, in order to use multiple frames of data from the tracks, the exact algorithm for multiframe test for the problem of T2TA is derived. The algorithm accounts for the crosscorrelations between tracks for the same target and the crosscorrelations across multiple frames of data. In practice, the T2TA algorithm is used for a sliding window of frames. The resulting sliding window test is shown to yield false rejections of H₀ (“the two tracks are from the same target”) that match the theoretical values. Then the sliding window test is compared with the single time test. Simulation results show that, counterintuitively, the sliding window test may have less power than the single time test. With an illustrative example, where two targets move in formation, namely, they keep a constant separation (their motion has the same process noise), it is shown that the power of the test depends not only on

⁴This is because with lower process noise the “memory” of the filter is “longer”.

Figure 4. Power of the test: sliding window test vs. single frame test with fusion a increased testing interval of 15 s
the number of frames of data, which determines the degrees of freedom of the test statistic, but also on the noncentrality parameter of the test statistic under $H_1$ (the two tracks are from different targets'). The results indicate that, when the multiple frames of data selected for T2TA are strongly correlated, which happens for motion with low process noise or the multiple frames of data are taken in a short time interval, the sliding window test may be counterproductive and has less power than the single time test. In practice, this should be avoided by, e.g., increasing the time difference between the selected data frames.

**APPENDIX – THE RECONSTRUCTION OF LOCAL INFORMATION AT THE FUSION CENTER**

Section II describes the exact sliding window test statistic for the T2TA. It is pointed out, when the local weight matrices (11)–(12) are not available at the fusion center, the direct transfer of these weights are too costly. Notice that these weights are functions of $F(i-1)$, $H_s(i)$, $K_s(i)$, $i = l + 1, \ldots, k$. At the fusion center, when estimates of the target positions and the locations of the local sensors are available, the measurement matrices used by the local sensor and the related updates done at the local tracker can be approximately replicated (using the fuser estimates, location of the local sensor and the time stamps when the measurements are taken), which yields $P_s(i)$, $K_s(i)$, $i = l + 1, \ldots, k$. Thus the approximate evaluation of the local weights (11)–(12) is given by

$$W_c^w(k, l) = \prod_{i=0}^{k-1} |I - K_s(i)\overline{H}_s(k - i)| F(k - i - 1)$$

(43)

$$W_u^w(k, i - 1) = -\prod_{j=0}^{k-1} |I - K_s(k - j)| F(k - j - 1)$$

(44)

$$W_v^w(k, i) = \prod_{j=0}^{k-1} |I - K_s(k - j)| F(k - j - 1)$$

(45)

For the T2TA, (14) can be approximated by

$$P_{cs}(k|k) = W_c^w(k, l)P_{cs}(l|l)W_c^w(k, l)^t + \sum_{i=l+1}^{k} W_c^w(k, i - 1)Q_q(i - 1)W_c^w(k, i - 1)^t$$

(46)

It is concluded that, to approximately calculate the weights (11)–(12) required in the sequential T2TA algorithm in Section II, the central tracker needs to know the positions of the local sensors and the time when the measurements are obtained. Although extra computations are required at the fusion center to obtain the approximate weights at the local trackers, these computations are affordable due to the simplicity of

3If the measurements are linear (as in [12]) there is no need for this approximation. In the problem considered later, the measurements are nonlinear (range and azimuth); consequently, this approximation will be needed.

In distributed tracking systems with sensors at fixed locations, there is no need to transfer the locations of local sensor to the fusion center. In a tracking system with sensors mounted on mobile platforms, the positions of these platforms are usually known at the fusion center, thus no extra burden in communication exists in transferring the locations of the local sensors. Table III compares the amount of data that needs to be transferred from a local tracker to the fusion center in the exact algorithm and the approximate algorithm.

Given that the transfer of measurement time stamps is much less expensive than the transfer of the local weights, namely, $(M + 1)\alpha^2 n_{acc} \gg Mn_T$, the savings in communication by the approximate algorithm is very significant.

### REFERENCES


