Abstract - Multi frequency lines, concurrently processed with bearing measurements, are widely used for passive sonar target motion analysis (TMA). This paper analyzes the impact of utilizing additional narrow band (NB) lines on the target range accuracy obtained after TMA. More specifically, the TMA range and velocity accuracy improvement that is obtained from adding multiple frequency lines is demonstrated mathematically. This is a new result, since to date this performance improvement has only been shown numerically via Monte Carlo simulations. Data recorded at sea is utilized to validate the results and demonstrate the improvement obtained by using additional frequency tracks in the TMA process. This is a good example of the benefit of the association and fusion of multiple frequency bearing tracks.

Keywords: TMA, Cramer Rao lower bound, passive sonar, narrow band, tracking, track-to-track association.

1 Introduction.

The most standard target motion analysis (TMA) consists in estimating target's position and (constant) velocity from bearings-only measurements corrupted by noise [1]. When frequency line is also available, own ship's maneuver is not necessary provided that the emitted and unknown frequency is a constant and the bearing rate is not equal to zero [5][16]. This well known method will be called in the sequel FBTMA. The fundamental question is then the following: is the improvement of the accuracy (in terms of covariance matrix) always guaranteed whereas the state dimension augments with the number of frequency lines?

In [2], a partial answer was given by Monte Carlo simulations, but not rigorously demonstrated mathematically.

\[1\] In the literature [2, 17], its name is also Doppler and bearing TMA.
track association algorithms ([11], [12]) are an important requirement for modern sonar systems.

3 Frequency and bearing target motion analysis (FBTMA)

3.1 Notation and hypotheses.

The observer (O) and the single source (S) are assumed traveling in the common plane at constant velocity and constant heading. Moreover, the source emits a single tone with constant and unknown frequency \( f_0 \). The state vector \( Y_S \) is hence defined as follows:

\[
Y_S = \left[ f_0, x_S(t^*), y_S(t^*), v_x, v_y \right]^T
\]

where \( f_0 \), \( x_S(t^*) \), \( y_S(t^*) \), \( v_x \), \( v_y \) are respectively the unknown emitted frequency, the components of the position at time \( t^* \) (the chosen reference time) and the components of the velocity of the source. For commodity reasons, we denote the last four components of \( Y_S \) by

\[
X_S = \left[ x_S(t^*), y_S(t^*), v_x, v_y \right]^T.
\]

So \( Y_S = [f_0, X_S]^T \). The noise-free azimuths are defined by

\[
\beta(X_S, t_k) = \text{atan} \left[ \frac{x_O(t_k) - x_O(t_k)}{y_O(t_k) - y_O(t_k)} \right]
\]

where \( x_O(t_k) \) and \( y_O(t_k) \) are the components of observer’s position at time \( t_k \).

The noise-free Doppler-shifted frequency is then

\[
f(t) = f_0 \left[ 1 - \frac{v(t)}{C} \right]
\]

where \( v(t) \) is the radial velocity and \( C \) is the sound speed in the medium (about 1500 m/s in water). The radial velocity must be expressed in terms of speed vector.

\[
v(X_S, t) = (v_x - v_{x_0}) \sin \left[ \beta(X_S, t) \right] + (v_y - v_{y_0}) \cos \left[ \beta(X_S, t) \right]
\]

where \( v_{x_0} \) and \( v_{y_0} \) are the components of observer’s speed vector.

Under those classical assumptions, the so-called frequencies and bearings target motion analysis (FBTMA) consists in estimating \( Y_S \) from a collection of couples of measurements.

\[
\left\{ \beta_m(t_k) = \beta(X_S, t_k) + \epsilon_{\beta}(t_k) \right\}
\]

\[
f_m(t_k) = f(Y_S, t_k) + \epsilon_f(t_k) \quad \text{for} \ (k = 1,\ldots, K)
\]

where \( \epsilon_{\beta}(t_k) \) et \( \epsilon_f(t_k) \) are the additive noise on bearings and frequencies, respectively. In the sequel, these noise are assumed independent, 0-mean Gaussian and their standard deviations are \( \sigma_{\beta}(t_k) \) et \( \sigma_f(t_k) \), respectively, assumed known (or previously estimated).

Remarks :

a) The azimuths are defined by \( X_S \) only while the frequency must be defined by \( Y_S \).

b) In a TMA context, \( f_0 \) has to be considered as a nuisance parameter, in the sense that we must estimate it but we are concerned essentially by the vector \( X_S \).

3.2 The FIM in the FBTMA

The Fisher information matrix (FIM) about \( Y_S \) takes the following form:

\[
F(Y_S|\beta_m, f_m) = \sum_{k=1}^{K} \frac{1}{\sigma_{\beta}(t_k)} \nabla_{\beta} \beta(X_S, t_k) \nabla_{\beta} \beta(X_S, t_k) + \sum_{k=1}^{K} \frac{1}{\sigma_f^2(t_k)} \nabla_{f} f(Y_S, t_k) \nabla_{f} f(Y_S, t_k)
\]

Due to the special structure of the state vector (2), the FIM (5) can be partitioned into four blocks:

\[
F(Y_S|\beta_m, f_m) = \begin{bmatrix} \gamma_0 & c_0^T \\ c_0 & F(X_S|\beta_m) + F(X_S|f_m) \end{bmatrix}
\]

with:

\[
F(X_S|\beta_m) = \sum_{k=1}^{K} \frac{1}{\sigma_{\beta}(t_k)} \nabla_{\beta} \beta(X_S, t_k) \nabla_{\beta} \beta(X_S, t_k)
\]

which is nothing else but the usual FIM about the state vector \( X_S \) given the bearing measurements.

The FIM about \( X_S \) given the frequency measurements is the following \( (4 \times 4) \) matrix

\[
F(X_S|f_m) = \sum_{k=1}^{K} \frac{1}{\sigma_f^2(t_k)} \nabla_{f} f(Y_S, t_k) \nabla_{f} f(Y_S, t_k)
\]

The last two blocks of (5) are the vector \( (4 \times 1) \),

\[
c_0 = \sum_{k=1}^{K} \frac{1}{\sigma_f^2(t_k)} \left[ \frac{\partial f(Y_S, t_k)}{\partial f_0} \right] \nabla_{f} f(Y_S, t_k)
\]
and the scalar
\[
\gamma_0 = \sum_{k=1}^{K} \frac{1}{\sigma_k^2(t_k)} \left( \frac{\partial f(Y_s, t_k)}{\partial f_0} \right)^2 \tag{9}
\]

We are going to exploit this decomposition of the FIM to prove a remarkable property about the accuracy of the TMA when the number of received frequencies increases.

4 Multi Frequency and bearing target motion analysis (MFBTMA)

4.1 Notations.

In this section, the source is supposed to emit \( N \) pure stable and unknown frequencies \( f_0, f_1, \cdots, f_{N-1} \).

The state vector (1) must hence be augmented as follows:
\[
\begin{bmatrix}
X_{10}^T \\
\vdots \\
X_{N0}^T
\end{bmatrix}
\]

The measurements collected by the observer is then \( \{ Y_m, f_0, \cdots, f_{N-1} \} \).

The \( N \) measured frequencies are given by
\[
f_{p,m}(t_k) = f_p(Y_s, t_k) + e_{f_p}(t_k),
\]
for \( p = 0,1,\cdots,N-1 \), with, as previously (4)
\[
f_p(Y_s, t_k) = f_p[1 - v(X_s, t_k)/C].
\]

The variance of \( e_{f_p}(t_k) \) is denoted \( \sigma_{f_p}(t_k) \).

The fundamental question of the MFBTMA is the following:

Does the MFBTMA potentially improve the accuracy of the estimate of \( X_s \)? In terms of Cramèr-Rao lower bound (CRLB), the question is
\[
B_{CR}(X_s|\beta_m, f_{0,m}, \cdots, f_{N-1,m}) 
\]

\[ \leq B_{CR}(X_s|\beta_m, f_0,m) \]  

Remarks:

a) We implicitly assume that there is one single bearing measurement, the same one for all the frequencies at each sampling time. In reality, a measured frequency is always linked to a measured bearing. Hence, we should have as many azimuths as frequencies. Because all these azimuths are relative to the same line of sight, they are averaged and the result is a more accurate single measured bearing. In order to make a fair comparison, the standard deviation of the bearing has been kept at the same value. So, the impact of the extra frequency lines will be objectively judged.

b) Again, the \( N \) unknown frequencies \( f_0, f_1, \cdots, f_{N-1} \) are nuisance parameters.

c) We insist on the fact that the bearing rate is not equal to zero and that the measurements are unbiased.

d) The numbering of the frequencies is arbitrary.

e) The association between traks is supposed error-free.

4.2 Expression of the FIM in a block structure

As previously (16), the FIM of MBFTMA can be partitioned as:
\[
F(Y_s|\beta_m, f_{0,m}, \cdots, f_{N-1,m}) = \begin{bmatrix}
\gamma_0 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \gamma_{N-1}
\end{bmatrix}
\begin{bmatrix}
c_0^T \\
\vdots \\
c_{N-1}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
F(X_s|\beta_m) + F(X_s|f_{0,m}) + \cdots + F(X_s|f_{N-1,m})
F(X_s|\beta_m) + F(X_s|f_{0,m}) + \cdots + F(X_s|f_{N-1,m})
\end{bmatrix}
\]

The vectors \( c_p \) and the scalars \( \gamma_p \) are similarly defined than \( c_0 \) (8) and \( \gamma_0 \) (9) respectively.

We are interested by the inverse of the submatrix \( F(X_s|\beta_m) + F(X_s|f_{0,m}) + \cdots + F(X_s|f_{N-1,m}) \) corresponding to the position and the velocity of the source.

We are going to use the following classic result of linear algebra (see [13]):

Let a non-singular matrix \( A \) and \( B \) its inverse, both partitioned in four blocks
\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\]

then \( B^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12} \) provided \( A_{11} \) is non-singular.

We then identify
rate are non null too and as a consequence the matrix $A_{11}$ is non singular. The recalled result can be applied.

We get $B_{22} = B_{CR}(X_s | \beta_m, f_{0,m}, \ldots, f_{N-1,m})$ given by

$$B_{CR}(X_s | \beta_m, f_{0,m}, \ldots, f_{N-1,m})^{-1} = \left[ F(X_s | \beta_m) + F(X_s | f_{0,m}) + \cdots + F(X_s | f_{N-1,m}) \right]^{-1}$$

or equivalently

$$B_{CR}(X_s | \beta_m, f_{0,m}, \ldots, f_{N-1,m})^{-1} \geq B_{CR}(X_s | \beta_m, f_{0,m}, \ldots, f_{N-1,m})^{-1}$$

We can re-write (11) as follows

$$B_{CR}(X_s | \beta_m, f_{0,m}, \ldots, f_{N-1,m})^{-1} =$$

$$\left[ F(X_s | \beta_m) + F(X_s | f_{0,m}) + \cdots + F(X_s | f_{N-1,m}) \right]^{-1}$$

The answer is unambiguous if the matrix $F(X_s | f_{N,m}) - \gamma_N^{-1} c N c_N^T$ is non negative. This property is satisfied since the matrix

$$F(X_s | f_{N,m}) - \gamma_N^{-1} c N c_N^T$$

is the Fisher information matrix $F(Y_s | f_{N,m})$, if we measure $f_{N,m}$ only. Hence, this matrix is a symmetric non negative matrix. The property (10) is then rigorously proven.

## 4.4 Algorithm aspects

The MFBTMA can be obtained via several estimators:

- The recursive estimators given by the pseudo linear filters [4], or by the extended Kalman filter [3] which all suffer from the same instability than in BOTMA, especially when the source is far from the CPA (Closest Point of Approach). The particular filters [15] could be used for a recursive estimation.

- The batch estimators, for example those given by the so-called MIV (for modified instrumental method) [2] [17], or the least squares estimator equivalent to the maximum likelihood estimator in the case of additive and Gaussian noise.

We suggest retaining the latter estimator which consists in minimizing the following quadratic criterion

$$\sum_{p=0}^{N-1} y_p^T c_p c_p^T$$
The estimate $\hat{Y}_k$ is reached with the Gauss-Newton procedure (or the Newton-Raphson one) demanding less than twelve iterations. The initialization point must be carefully chosen (or guessed): according to the situation, we can benefit from prior information or we can use a coarse estimate chosen among the nodes of an ad-hoc grid.

We recommend to employ the so-called pseudo-linear estimator (given by the first iteration of the MIV), as suggested in [2]. Concerning the frequencies, we propose to use the mean value of each track as initial value of the emitted frequency.

$$C(Y_S) = \sum_{k=1}^{K} \left[ \frac{\beta_p(t_k) - \beta(X_S, t_k)}{\sigma_p(t_k)} \right]^2 + \sum_{p=0}^{N-1} \sum_{k=1}^{K} \left[ \frac{f_{p,n}(t_k) - f_p(Y_S, t_k)}{\sigma_{f_p}(t_k)} \right]^2.$$  

The obtained results were exploitable after the 26th minute, before the weak observer maneuver which occurred at the 32nd minute.

The relative error of the estimated range is less than 20% at the 30th minute and less than 10% at the final instant.

5.3 Comparison between FBTMA and MFBTMA at sea

With each frequency line, we ran the FBTMA with the batch algorithm in the same previous conditions. We obtained seven estimated range history displayed in Fig.5. The result are disperse and the error can reach 30 to 60% at the 30th minute depending on the chosen frequency line. The same report can be made at the final time.

For instance, with the frequency line around 90Hz or by around 110Hz, the error in the estimated range can reach 20% at the final time, whereas with these close to 97Hz, the result is excellent. We can conclude that the accuracy of the FBTMA depends hence on the chosen frequency line and so not very reliable. This is likely due to some bias of the measurement lines.

If these seven FBTMA results are averaged, we obtain the dash line of the fig. 5 very close to the similar curve displayed in fig. 4 concerning the MFBTMA.

The advantage of the MFBTMA on the FBTMA is that a sole run of the ad-hoc algorithm provides a reliable solution while the FBTMA demands as many runs as frequency lines to get the same result. This point disqualifies the use of the FBTMA in case of a large number $N \geq 100$ of frequency lines.

6 Conclusion

In passive narrow band sonar, it is usual to track targets emitting several pure single tones. At each sampling time, the available measurements are a bearing and a set of frequencies. The crucial question we answered was to know if it’s worth using all these frequencies in a TMA function, since each frequency augments the dimension of the state vector.

We have rigorously demonstrated that the more frequencies are taken into account in the TMA, the more accurate the target trajectory estimate (positions and speeds) will be: it is a theoretical point obtained by the computation of the CRLB.

The confrontation to at sea data confirms that fact: more precisely, we observe shorter convergence of the algorithm and better accuracy on the estimated target range (or conversely a better accuracy within the same time). The same approach can be conducted in the case of frequency comb, i.e. harmonically separated frequency lines.
7 References


Figure 1: Lofargram with interferences and chosen frequency lines (green markers)
Figure 2: Scenario used for the comparison

Figure 3: Normalized residuals vs. time in min. (vertical axis) of the estimates returned by the MFBTMA.
Figure 4: MFBTMA estimated range and real range between the 20th and the 56th minute.

Figure 5: Several Single FB TMA results and averaged result.