Relation of Optimal Local Compression and Local Likelihood Ratio

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Abstract—As Tenney and Sandell pointed out, the optimal local compression/decision rule has the form of a likelihood ratio when the local observations are not correlated [3]. However, this does not hold in general for correlated observations [4]. An interesting problem is to find the conditions under which the optimal local compression rule remains to have the form of a likelihood ratio even for correlated observations. In this paper, we prove that, with the model of Gaussian signal with independent Gaussian noise, the optimal local compression rule has the expected likelihood ratio form when the two conditional probability density functions are centrosymmetric. Computer simulation is provided in the paper for demonstration.

I. INTRODUCTION

In recent years, data fusion or information fusion techniques have been widely applied in various application areas, such as target tracking, image processing, economic data analysis, etc. The motivation behind using multiple sensors has many folds, for example to reduce error and uncertainty in the measurement, to obtain results about the measurement that would not be accessible using a single sensor, etc.

There are two basic fusion architectures, centralized and decentralized/distributed, depending on whether raw data are sent to the fusion center or not. For centralized fusion, all the sensor signals are implicitly assumed to be available in one place for processing. The centralized sensor signal processing is based on the statistical estimation and hypothesis testing method. The situation is substantially more complicated in the case of a distributed sensor system where each sensor only sends out locally processed information due to various constrains, such as reliability, survivability, communication bandwidth, or even simply the problem of flooding the fusion center with more information than it can process. The interest of the distributed sensor systems has been sparked by the requirements of military surveillance systems and is reflected in the widespread use of such terms as data fusion, correlation, and multi-sensor integration [3]. Distributed fusion is more challenging and has been a focal point in fusion research for many years.

Consider a decision system with a distributed sensor network. Each local sensor observes data and may receive messages from other sensors simultaneously. It locally fuses/compresses all its available information to a communicable message and transmitted it to a fusion center. The fusion center makes a final decision using all received messages with some fusion rule. Communications are possibly permitted not only between the sensors and the fusion center, but also among sensors themselves.

In the distributed decision system, the fusion rule and the set of local sensor decision rules should be tightly coupled in order to globally optimize the system performance given a communication pattern of the system. For distributed decision, there are two classical approaches. In the first approach, the fusion rule is fixed [3]. On the other hand, the local sensor decision rules are fixed in the second case [4], [1], [2]. Recently, some new ideas had been proposed to achieve a quite general distributed decision system [9], [6], [7], in which finding the optimum fusion rule is reduced to determining the sensor rules that yield optimum system performance.

For the first approach, Tenney and Sandell pointed out, the optimal local compression/decision rule has the form of a likelihood ratio when the local observations are not correlated [3]. However, this does not hold in general for correlated observations. In this paper, we prove that, with the model of Gaussian signal with independent Gaussian noise, the optimal local compression rule has the expected likelihood ratio form when the two conditional probability density functions are centrosymmetric.

The rest of the paper is organized as follows. Section II presents the problem formulation. In Sections III, IV, and V, we prove the main results of the paper that, for Gaussian signal with independent Gaussian noise, the optimal local compression rule has the expected likelihood ratio form when the two conditional probability density functions are centrosymmetric. In Section VI, we show the computer simulation for demonstration. Finally, Section VII concludes the paper.

II. PROBLEM FORMULATION

Consider a distributed decision problem of 2 hypotheses $H_0$, $H_1$ and 2 sensors with observation data $x$ and $y$. Each sensor makes a 2-ary decision $u_x$ and $u_y$ based on the local information and then transmits its decision out. Then, a fusion center makes a final 2-ary decision based upon all the received messages of local sensor decisions. The structure of this distributed fusion is shown in Fig. 1.
With Gaussian noises, the nonrandomized fusion rule $F$ given by the known conditional probability density functions of the observations under the two hypotheses, respectively. Let $\{u_x, u_y\}$ be the observations of the fusion center, where $u_x$ and $u_y$ are equal to 0 or 1 corresponding to hypothesis $H_0$ or $H_1$ that is declared to be detected, respectively.

The fusion criterion is a function

$$J : \{0, 1\} \times \{0, 1\} \times \{H_0, H_1\} \rightarrow \mathbb{R}$$

with $J(u_x, u_y, H_i)$ being the cost incurred for sensor $x$ choosing $u_x$, and sensor $y$ choosing $u_y$, when $H_i$ is true.

Specifically, we consider a two-dimensional Gaussian signal $s$ with Gaussian noises $n_1$ on $x$-axis observed by sensor $x$ and $n_2$ on $y$-axis observed sensor $y$:

$$s \sim \mathcal{N}\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right)$$

$$n_1 \sim \mathcal{N}(0, \sigma_1^2), \quad n_2 \sim \mathcal{N}(0, \sigma_2^2)$$

$$E(n_1 n_2) = \delta$$

When the signal and noises are all independent, the two conditional probability density functions are as follows:

$$P(x, y|H_0) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right)$$

$$P(x, y|H_1) \sim \mathcal{N}\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma_1^2 + \sigma_1^2 & \sigma_2^2 \\ \sigma_1^2 & \sigma_2^2 + \sigma_2^2 \end{bmatrix}\right)$$

where

$$H_0 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$H_1 : \begin{bmatrix} x \\ y \end{bmatrix} = s + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

Once $\mu = 0$, the two conditional probability density functions are obviously centrosymmetric, i.e., $f(x) = f(-x)$, and their difference with weight $\lambda$ ($0 \leq \lambda \leq 1$)

$$L(x, y) = P(x, y|H_1) - \lambda P(x, y|H_0)$$

is also centrosymmetric.

For distributed decision fusion, the optimal local decision intervals are

$$A = \{x|\mathcal{G}_1(x) \leq 0\} \quad B = \{y|\mathcal{G}_2(y) \leq 0\}$$

where $\mathcal{G}_1(x) = \int_{\mathcal{G} \times \mathcal{G}} L(x, y) dy$, $\mathcal{G}_2(y) = \int_{\mathcal{G} \times \mathcal{G}} L(x, y) dx$.

It is well known that, given a fusion rule, the distributed decision region determined by optimal local compression rule is centrosymmetric. And the decision region is constituted by some rectangular regions $A \times B$:

$$A = \bigcup_{i=1}^{l_1}[a_i, b_i]\quad B = \bigcup_{i=1}^{l_2}[c_i, d_i]$$

where

$$[a_i, b_i] \cap [a_j, b_j] = \emptyset \quad [c_i, d_i] \cap [c_j, d_j] = \emptyset$$

for all $i \neq j$.

More generally, when the means of two conditional probability density functions are equal, we have the same result since we can move their symmetric centers to the origin. Moreover, it holds for any data with a symmetric distribution.

In the next two sections, we will prove that, for the above model of a Gaussian signal with independent Gaussian noises, the optimal local compression rule has the form of a likelihood ratio for correlated local observations under the following conditions 1) only one rectangle constitutes the decision region under optimal local compression rule; 2) the function $L(x, y)$ is centrosymmetric, or by parallel-moving, it can become centrosymmetric.

### III. Three Types of $\lambda$

To make the discussion more convenient, we denote

$$\Delta_1 = (\sigma_1^2 + \sigma_2^2)\sigma_2^2 + \sigma_1^2 \sigma_2^2$$

$$\Delta_2 = \sigma_1^2 \sigma_2^2$$

Let

$$F(x, y) = P(x, y|H_1)$$

$$G(x, y) = P(x, y|H_0)$$

we have

$$F(x, y) = \frac{1}{2\pi\sqrt{\Delta_1}} e^{-\frac{x^2 + y^2}{2\Delta_1} + \frac{\Delta_1 x^2}{\Delta_1}}$$

$$G(x, y) = \frac{1}{2\pi\sqrt{\Delta_2}} e^{-\frac{x^2 + y^2}{2\Delta_2} + \frac{\Delta_2 x^2}{\Delta_2}}$$

Now, we define

$$\Omega^- = \{(x, y)|L(x, y) \leq 0\}$$

Obviously, $(A \times B) \cap \Omega^- \neq \emptyset$ if $(A \times B) \neq \emptyset$ and $\Omega^- \neq \emptyset$. 

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Fig. 1. Distributed Decision Fusion.
Let
\[ T(x, y) = \ln \frac{F(x, y)}{\lambda G(x, y)} = \frac{\sigma^2 y + \sigma^2 \sigma x}{2\Delta_1 \Delta_2} + \frac{1}{2} \ln \frac{\Delta_2}{\Delta_1} - \ln \lambda \]

Then, \( \Omega^- = \{(x, y) | T(x, y) \leq 0\} \).

There are three possible ranges of \( \lambda \):

1) \( \Lambda_1 = \{\lambda | \ln \lambda - \ln \sqrt{\frac{2\lambda}{\Delta_1}} < 0\} \)
   - For \( \lambda \in \Lambda_1 \), we have \( \Omega^- = \emptyset \) and thus \( A \times B = \emptyset \).

2) \( \Lambda_2 = \{\lambda | \ln \lambda - \ln \sqrt{\frac{2\lambda}{\Delta_1}} = 0\} \)
   - For \( \lambda \in \Lambda_2 \), it can be shown that \( y = -\frac{\sigma^2}{\sigma^2 t} x \). Thus, \( \Omega^- = \{(x, -\frac{\sigma^2}{\sigma^2 t} x) | x \in \mathbb{R}\} \) and \( A \times B \) has only one point \((0, 0)\). Besides, \( \int_{A \times B} L(x, y) dy dx = 0 \).

3) \( \Lambda_3 = \{\lambda | \ln \lambda - \ln \sqrt{\frac{2\lambda}{\Delta_1}} > 0\} \)
   - For \( \lambda \in \Lambda_3 \), \( \Omega^- = \{(x, y) | \omega \leq \frac{\sigma^2 y + \sigma^2 \sigma x}{\Delta_2}\} \), where \( \omega = \sqrt{(\ln \lambda - \frac{1}{2} \ln \frac{\Delta_2}{\Delta_1}) (2\Delta_1 \Delta_2)} \).

Since the first two cases are trivial, we will discuss only the third case in detail in what follows.

Consider any extreme point \((x_0, y_0)\) in \( \Omega^- \), then
\[ L_x(x_0, y_0) = 0 \quad (1) \]
\[ L_y(x_0, y_0) = 0 \quad (2) \]
\[ T(x_0, y_0) \leq 0 \quad (3) \]

Summing Equations (1) and (2), we obtain
\[ (-\sigma^2 x_0 - \sigma^2 y_0) (\frac{F}{\Delta_1} - \frac{\lambda G}{\Delta_2}) = 0 \quad (4) \]
which indicates either \( \frac{F}{\Delta_1} = \frac{\lambda G}{\Delta_2} \) or \( \sigma^2 x_0 + \sigma^2 y_0 = 0 \).

If \( \frac{F}{\Delta_1} = \frac{\lambda G}{\Delta_2} \), it is easy to show
\[ T(x_0, y_0) + \ln \frac{\Delta_2}{\Delta_1} = 0 \quad (5) \]

However, Equation (5) cannot hold because \( T(x_0, y_0) \leq 0 \) and \( \ln \frac{\Delta_2}{\Delta_1} < 0 \).

If \( \sigma^2 x_0 + \sigma^2 y_0 = 0 \), we may get
\[ T(x_0, y_0) = \ln \frac{\frac{\sigma^2 x_0}{\Delta_1 \Delta_2}}{\frac{\sigma^2 y_0}{\Delta_1 \Delta_2} - \frac{\sigma^2}{\Delta_1} y_0} \]
based on Equation (1) if the denominator \( \frac{\sigma^2 + \sigma^2 x_0 - \sigma^2 y_0}{\Delta_1} \neq 0 \). Since \( T(x_0, y_0) < 0 \) in this case, we can derive \( \sigma^2 x_0 + \sigma^2 y_0 < 0 \), which contradicts the condition.

Based on the above discussion, we obtain that any point \((x_0, y_0)\) satisfying Equations (1), (2), and (3) in \( \Omega^- \) also satisfies
\[ \frac{\sigma^2 + \sigma^2}{\Delta_1} x_0 - \frac{\sigma^2}{\Delta_1} y_0 = 0 \]
\[ \sigma^2 x_0 + \sigma^2 y_0 = 0 \]

which result in that \((x_0, y_0)\) has to be \((0, 0)\). Consider the second order derivatives
\[ L_{xx}(0, 0) = -\frac{\sigma^2 + \sigma^2}{\Delta_1} + \frac{\sigma^2}{\Delta_2} \lambda \]
\[ L_{yy}(0, 0) = -\frac{\sigma^2 + \sigma^2}{\Delta_1} + \frac{\sigma^2}{\Delta_2} \lambda \]
\[ L_{xy}(0, 0) = L_{yx}(0, 0) = \frac{\sigma^2}{\Delta_1} \lambda \]

Since \( T(0, 0) < 0 \) and \( \lambda > \sqrt{\frac{2\lambda}{\Delta_1}} \), we have
\[ L_{xx}(0, 0) > \frac{\sigma^2}{\Delta_1 \Delta_2} \frac{1}{2\sqrt{\Delta_1}} > 0 \]
\[ L_{yy}(0, 0) > \frac{\sigma^2}{\Delta_1 \Delta_2} \frac{1}{2\sqrt{\Delta_1}} > 0 \]

Hence, the Hessian matrix at \((0, 0)\) is positive definite
\[ H = \begin{bmatrix} L_{xx}(0, 0) & L_{xy}(0, 0) \\ L_{yx}(0, 0) & L_{yy}(0, 0) \end{bmatrix} > 0 \]
Thus, the point \((0, 0)\) is a local minimum of \( L \). Also \((0, 0)\) is the only stationary point in \( \Omega^- \).

IV. THE CONSTITUTION OF THE OPTIMAL DISTRIBUTED DECISION REGION

Consider the local decision interval \( A \)
\[ A = \{x | \int_B L(x, y) dy \leq 0\} \]
\[ = \{x | \int_B F dy \leq \lambda\} \]
\[ = \{x | \ln \int_B F dy \leq \ln \lambda\} \]
Let \( \Pi(x) = \ln \int_B F dy \int_B G dy - \ln \lambda, A = \{x | \Pi(x) \leq 0\} \). Now
\[ \Pi'(x) = \frac{\sigma^2}{\Delta_1 \Delta_2} x + \frac{\sigma^2}{\Delta_1} \int_B y F dy \]
\[ \Pi''(x) = \frac{\sigma^2}{\Delta_1 \Delta_2} + \frac{\sigma^2}{\Delta_1} \int_B \left( -\frac{\sigma^2 + \sigma^2}{\Delta_1} x + \frac{\sigma^2}{\Delta_1} y \right) y F dy \]
\[ - \int_B y F dy \int_B \left( -\frac{\sigma^2 x_0 + \sigma^2 y_0}{\Delta_1 \Delta_2} - \frac{\sigma^2}{\Delta_1} y_0 \right) F dy \]
Let \( P_x(y) = \frac{F}{\int_B F dy} \). Given an \( x \), \( P_x(y) \) is the probability density function on \( B \). We have
\[ \Pi''(x) = \frac{\sigma^2}{\Delta_1 \Delta_2} + \frac{\sigma^2}{\Delta_1} \left( \int_B (y F dy - (E_y F dy) F dy \right) \]
\[ = \frac{\sigma^2}{\Delta_1 \Delta_2} + \left( \frac{\sigma^2}{\Delta_1} \right)^2 Var_p(y) \]
Obviously, \( \Pi''(x) > 0 \). Thus, \( \Pi(x) \) is strictly monotonously increasing.

Moreover, we can prove \( \Pi(a_i) = \Pi(b_i) = 0, i = 1, 2, \ldots, l_1 \) as shown in Appendix I. Thus, there is an
(a_i, b_i) satisfying \( \Pi'(x_i) = 0, i = 1, 2, \ldots , l_1 \) according to the mean value theorem. Because \( \Pi'(x) \) is strictly monotonously increasing, \( l_1 \) must be one. So \( A = [a, b] \). Likewise \( B = [c, d] \).

Now, we may conclude that the optimal distributed decision region is constituted by only one rectangular area.

V. Centrosymmetric Property of the Optimal Distributed Decision Region

Let

\[
U(z, w, p, q) = \int_z^w \int_p^q L(x, y) dy dx
\]

then

\[
f_1(z, w, p, q) = U'_z(z, w, p, q) = \int_p^q L(z, y) dy
\]

\[
f_2(z, w, p, q) = U'_w(z, w, p, q) = \int_p^q L(w, y) dy
\]

\[
f_3(z, w, p, q) = U'_p(z, w, p, q) = \int_z^w L(x, p) dx
\]

\[
f_4(z, w, p, q) = U'_q(z, w, p, q) = \int_z^w L(x, q) dx
\]

since the the optimal \((a, b, c,d)\) is the minimizer of \(U(\cdot)\), it should satisfy

\[
f_1(a, b, c, d) = \int_c^d L(a, y) dy = 0 \quad (6)
\]

\[
f_2(a, b, c, d) = \int_c^d L(b, y) dy = 0 \quad (7)
\]

\[
f_3(a, b, c, d) = \int_a^b L(x, c) dx = 0 \quad (8)
\]

\[
f_4(a, b, c, d) = \int_a^b L(x, d) dx = 0 \quad (9)
\]

In what follows, we discuss the relations between \(a, b, c\) and \(d\).

A. Case I

If \(z\) and \(w\), \(p\) and \(q\) are not related, i.e. \(w\) can not be expressed by \(z\), and \(q\) can not be expressed by \(p\), then

\[
J_1 = \frac{\partial(f_1, f_2)}{\partial(z, w)}(a, b, c, d) = \begin{bmatrix}
\int_c^d L_x(a, y) dy & 0 \\
0 & \int_c^d L_x(b, y) dy
\end{bmatrix} \neq 0 \quad (10)
\]

\[
J_2 = \frac{\partial(f_3, f_4)}{\partial(p, q)}(a, b, c, d) = \begin{bmatrix}
\int_a^b L_y(x, c) dx & 0 \\
0 & \int_a^b L_y(x, d) dx
\end{bmatrix} \neq 0 \quad (11)
\]

A proof of (10) and (11) is given in Appendix II. With the implicit function theorem, there are four functions \(u, v\) over the field \(O(c, d)\), \(uu, vv\) over the field \(O(a, b)\) that satisfy

\[
z = u(p, q) \quad p = uu(z, w) \quad w = v(p, q) \quad q = vv(z, w)
\]

and

\[
a = u(c, d) \quad c = uu(a, b) \quad b = v(c, d) \quad d = vv(a, b)
\]

It can be shown that \(\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} = \frac{\partial u}{\partial w} \frac{\partial uu}{\partial z} = \frac{\partial uu}{\partial z} = 1\). Likewise \(\frac{\partial uu}{\partial q} \frac{\partial vv}{\partial z} = 1\). Thus,

\[
\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial w} \frac{\partial uu}{\partial z} = 2 \quad (12)
\]

For \(z = u(p, q)\), we have \(\frac{\partial z}{\partial q} = \frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z}\), i.e.,

\[
\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} = 1 \quad (13)
\]

Obviously, Equations (12) and (13) are not compatible, which implies that there are no non-related \(a, b, c, d\) satisfying Equations (6)-(9).

B. Case II

If \(w\) can be expressed as \(f(z)\), and \(p\) and \(q\) are independent, then

\[
J_1 = f'_z(a, c, d) = \int_c^d L_z(a, y) dy \neq 0
\]

\[
J_2 = \frac{\partial(f_3, f_4)}{\partial(p, q)}(a, c, d) = \begin{bmatrix}
\int_a^f(a) L_y(x, c) dx & 0 \\
0 & \int_a^f(a) L_y(x, d) dx
\end{bmatrix} \neq 0
\]

Hence, there must be one function \(u\) over the field \(O(c, d)\) and \(uu, vv\) over the field \(O(a)\) that satisfy

\[
z = u(p, q) \quad p = uu(z, w) \quad w = v(p, q) \quad q = vv(z, w)
\]

\[
a = u(c, d) \quad c = uu(a, b) \quad b = v(c, d) \quad d = vv(a, b)
\]

Same as above, we have \(\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} = 1\) since \(\frac{\partial z}{\partial p} = 1\) and \(\frac{\partial u}{\partial q} \frac{\partial q}{\partial z} = 1\) since \(\frac{\partial z}{\partial q} = 1\). So,

\[
\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} = 2 \quad (14)
\]

For \(z = u(p, q)\), we have \(\frac{\partial z}{\partial q} = \frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z}\), i.e.,

\[
\frac{\partial u}{\partial p} \frac{\partial uu}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} = 1 \quad (15)
\]

Obviously, (14) and (15) are incompatible. It implies that there are not any \(a, b = f(a), c\) and \(d\) which satisfy Equations (6)-(9).

C. Case III

If \(q\) can be expressed as \(g(p)\) and \(z\) and \(w\) are independent. Similar to case (2), there are no \(a, b, c, d = g(c)\) which satisfy Equations (6)-(9).
D. Case IV

If $w$ can be expressed as $f(z)$ and $q$ can be expressed as $g(p)$, then

$$f_1'(a,c) = \int_c g(c) L_x(a,y)dy \neq 0 \quad (16)$$

$$f_3'(a,c) = \int_a f(a) L_y(x,c)dx \neq 0 \quad (17)$$

According to the implicit function theorem, there is the function $u$ in the field $O(c)$, $v$ in the field $O(a)$ that satisfy

$$z = u(p) \quad a = u(c)$$
$$p = v(z) \quad c = v(a)$$

Thus, $u'(p)v'(z) = 1$, i.e. $\frac{f_1}{f_1} \frac{f_2}{f_3} = 1$. Therefore, in the field $O(a) \times O(c)$

$$\frac{D(f_1, f_3)}{D(z,p)} = \begin{bmatrix} f_{1z} & f_{1p} \\ f_{3z} & f_{3p} \end{bmatrix} = 0$$

According to the correlative function theorem, there must exist a function $\phi$ in the field $O(a) \times O(c)$ that satisfies $f_1(z,p) = \phi(f_3(z,p))$. Because $f_1(a,c) = \phi(f_3(a,c))$, we have $\phi(0) = 0$. It is known that for every $z \in O(a)$, there exists a $p = v(z)$ satisfying $f_3(z,p) = 0$. Hence, we can deduce that $f_1(z,p) = 0$. Since $L(x,y)$ is a centrosymmetric function, we can easily obtain $f(a) = -a$ and $g(c) = -c$ that satisfy the above conditions. Now, when Equations (6) and (8) hold, the equivalence of Equations (7) and (9) are the corresponding results. In this case, there is no contradiction any more as shown in Appendix III. From Section III, we know that $(0,0)$ is the only point in $\Omega^-$. Thus, if we begin from $(0,0)$ and enlarge the region little by little, we may get the optimal compression rule. Hence, in order to find the optimal local compression rule, we need only obtain the appropriate $a$ and $c$ ($a > 0$) that satisfy the following equations:

$$f_1(a,c) = \int_c L(a,y)dy = 0 \quad (18)$$
$$f_2(a,c) = \int_a L(x,c)dx = 0 \quad (19)$$

Now, the centrosymmetric distributed decision region is obtained. It can also be achieved by the likelihood ratio compression rule. Thus, the optimal local compression rule has the form of likelihood ratio test.

VI. SIMULATION

In this section, we consider the OR fusion rule, i.e.,

$$F(u_x = 1, u_y = 1) = 1$$
$$F(u_x = 1, u_y = 0) = 1$$
$$F(u_x = 0, u_y = 1) = 1$$
$$F(u_x = 0, u_y = 0) = 0$$

The two hypothesis are

$$H_0 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$
$$H_1 : \begin{bmatrix} x \\ y \end{bmatrix} = s + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where the signal $s$ and the two sensor observation noises $n_1$ and $n_2$ are Gaussian and all mutually independent

$$s \sim N(0,1), \ n_1 \sim N(0,0.1), \ n_2 \sim N(0,0.1)$$

Thus, the two conditional pdfs under $H_0$ and $H_1$, respectively, are

$$p(x,y|H_0) \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \right)$$
$$p(x,y|H_0) \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.1 & 1 \\ 1 & 1.1 \end{bmatrix} \right)$$
In the simulation, $\lambda$ is set to 1. In Fig. 2, we can find the decision region under optimal local compression rule is centrosymmetric, and is constituted with only one rectangular. In Fig. 3, the contour of $L(x, y)$ shows that there is only one stationary point $(0, 0)$ in $\Omega^\prime$.

VII. Conclusions

In this paper, we prove that under certain conditions, the optimal local compression rule has the form of likelihood ratio compression for correlated local observations. With the model of Gaussian signal with independent Gaussian noises, the conditions are 1) only one rectangular constitutes the decision region under optimal local compression rule; 2) the function $L(x, y)$ is centrosymmetric, or by parallel moving, it can become centrosymmetric. The proof can also be easily extended into a more general situation. However, the result that $(0, 0)$ is the only point in $\Omega^\prime$ holds only for the Gaussian model.

REFERENCES


APPENDIX I

$\Pi(a_i) = \Pi(b_i) = 0, i = 1, 2, \cdots, l_1$.

Proof: Because $a_i \in A, \Pi(a_i) \leq 0$. If $\Pi(a_i) < 0$, there is an $\epsilon > 0$ satisfying $\Pi(x) \leq 0, \forall x \in [a_i - \epsilon, b_i]$ according to the continuity of $\Pi(x)$. So $[a_i - \epsilon, b_i] \subseteq A$. But $[a_i - \epsilon, b_i] \cap [a_i, b_i] \neq \emptyset$, which is a contradiction. Thus, $\Pi(a_i) = 0$. Similarly, $\Pi(b_i) = 0$.

APPENDIX II

$\int_{-\epsilon}^{\epsilon} L_x(a, y) dy < 0, \int_{-\epsilon}^{\epsilon} L_y(x, c) dx < 0, \int_{-\epsilon}^{\epsilon} L_x(b, y) dy > 0,$ and $\int_{-\epsilon}^{\epsilon} L_y(x, d) dx > 0$

Proof: Denote $\Psi(x) = \int_{-\epsilon}^{\epsilon} L(x, y) dy$. Obviously, $\Psi(a) = 0$ and $\Psi(b) = 0$. According to the mean value theorem, it is easy to verify that there is an $x_0 \in (a, b)$ satisfying $\Psi'(x_0) = 0$. Besides,

$$\Psi''(x) = \int_{-\epsilon}^{\epsilon} L_x(x, y) dy$$

$$\geq \int_{-\epsilon}^{\epsilon} \left(-\frac{\sigma_2^2 + \sigma_2^2}{\Delta_1} F + \frac{\sigma_2^2}{\Delta_2} \lambda G + (\frac{\sigma_2^2}{\Delta_2} x^2) \lambda G dyight)$$

$$\geq \int_{-\epsilon}^{\epsilon} \left(-\frac{\sigma_2^2 + \sigma_2^2}{\Delta_1} F + (\frac{\sigma_2^2}{\Delta_2} x^2) \lambda G dyight) > 0$$

The above inequality is supported by $\int_{-\epsilon}^{\epsilon} [F(x, y) - \lambda G(x, y)] dy \leq 0$ for every $x \in [a, b]$, i.e. $\int_{-\epsilon}^{\epsilon} F(x, y) dy \leq \int_{-\epsilon}^{\epsilon} \lambda G(x, y) dy$. So $\Psi'(x)$ is strictly monotonously increasing in $(a, b)$. Therefore $\Psi'(a) < \Psi'(x_0) = 0$ and $\Psi'(b) < \Psi'(x_0) = 0$, i.e. $\int_{-\epsilon}^{\epsilon} L_x(a, y) dy < 0$ and $\int_{-\epsilon}^{\epsilon} L_x(b, y) dy > 0$.

APPENDIX III

There is no contradiction in Equations (6)-(9), when $a = -b$ and $c = -d$.

Proof: When $a = -b$ and $c = -d$, the equivalences of Equations (6)-(9) can be written as Equations (14) and (15):

$$f_1(a, c) = \int_{-c}^{c} L(a, y) dy = 0$$

$$f_2(a, c) = \int_{-a}^{a} L(x, c) dx = 0$$

$$f'_{1a}(a, c) = \int_{-c}^{c} L_x(a, y) dy < 0$$

$$f'_{2a}(a, c) = \int_{-a}^{a} L_y(x, c) dx < 0$$

Thus, there must be two only functions $u$ and $v$ that satisfy $a = u(c), c = v(a)$ and

$$u'(c) = -\frac{df_1}{dc} / f_{1a} = (-L(a, c) - L(-a, c))/f_{1a}$$

$$v'(a) = -\frac{df_2}{da} / f_{1c} = (-L(a, c) - L(-a, c))/f_{1c}$$

For $\frac{df_1}{dc} = 1, \frac{df_2}{da} = 1$ and $\frac{df_1}{dc} \frac{df_2}{da} = \frac{df_1}{dc} \frac{df_2}{da}$, so $\frac{df_1}{dc} \frac{df_2}{da} = 1$. The above two equivalences have no contradiction.