A Generalized Belief Fusion Algorithm

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Abstract – For a given body of possible belief evidential data, a processing objective is to glean the most correct knowledge from such data (but not more). The author introduced a new methodology of combining independent multi-source beliefs in 2002. This generalized belief fusion algorithm, depending on the probability proportionality weighting functions used, is shown to be equivalent, in the probability estimate, to the Dempster–Shafer (D-S) theory of evidence, the Fixsen-Mahler Modified Dempster–Shafer (MDS) theory and other new fusion methodologies that will converge faster to correct results. A more computationally friendly representation of the generalized belief fusion algorithm is given. A numerical example comparing five belief fusion variants is presented for the same inputs. This article also introduces a pignistic probability transform proportional to the normalized plausibility transform.

Keywords: Pignistic Probability Transform, Generalized Belief Fusion.

1. Introduction

Many variations of Belief Theories such as Dempster–Shafer (D-S) theory of evidence [1], the Fixsen-Mahler Modified Dempster–Shafer (MDS) theory [2,4,5] and Dezert-Smarandache Theory (DSmT) [9] have demonstrated practical and realistic methods of handling the many aspects of uncertain, indeterminate, conflicting, and shifting information in real fusion systems. This article outlines the Generalized Belief Fusion [7] methodology for making faster and better decisions using Belief function theories. This methodology can be used to automate critical decisions in complex systems.

Let $\Omega$ be the set of possible outcomes [6], where the outcomes are mutually exclusive and exhaustive singleton elements of the decision environment. In Bayesian formalism, the probabilities are assigned only to the singleton subsets of the quantitative information set. These probabilities are used to make the decisions. For systems with a complex input (real-time sensor measurements, multidimensional filtered feature extractions, real-time data base and a priori data base information content, natural language text and symbols parsing evidence, quantitative and qualitative communication clues, and inconsistent errors), a Power set $(\Omega) = 2^\Omega$ representation of the outcomes and a two-level (lower & upper) probability portrayal is a better representation of the incomplete information set; i.e., some sensor measurements of attributes populate more than one hypothesis. Belief theories maintain a two-level probabilistic portrayal of the information set: the Belief or credal level and the Plausibility level. The primary foundation in any decision proposition $A_j$ is the value of the Belief $Bel(A_j)$, while the plausibility $Pl(A_j)$ provides the secondary support for the decision. The basic belief assignment (BBA) $m(A_j)$ represents the strength of all the incomplete information set for the outcome $A_j$. The assignments of BBAs values to subsets $A_j \in 2^\Omega$ are constrained by the normalization constraint equation.

$$\sum_{A_j \in 2^\Omega} m(A_j) = 1$$  \hspace{1cm} (1)

Using the BBAs as the representative of the incomplete information set, the Belief function can be computed. The Belief of $A_j$ is the sum of all $m(A_k)$ for subsets $A_k$ contained in $A_j$:

$$Bel(A_j) = \sum_{A_k \supseteq A_j} m(A_k)$$  \hspace{1cm} (2)

The Plausibility of the subset $A_j$ is the sum $m(A_k)$ for all subsets $A_k$ that have a non-null intersection with $A_j$. It is given by:

$$Pl(A_j) = \sum_{A_k \cap A_j \neq \emptyset} m(A_k)$$  \hspace{1cm} (3)

A useful notation to simplify some derivations is the compound-to-sum operator, $\tilde{C}^S$ [6], which operates on singleton elements with the following properties:

$$\tilde{C}^S[Bel([A_j, A_1, ..., A_n])] = Bel(A_j) + Bel(A_1) + ... + Bel(A_n)$$

$$\tilde{C}^S[Pl([A_j, A_1, ..., A_n])] = Pl(A_j) + Pl(A_1) + ... + Pl(A_n)$$

$$\tilde{C}^S[m([A_j, A_1, ..., A_n])] = m(A_j) + m(A_1) + ... + m(A_n)$$  \hspace{1cm} (4)
2. Generalized Belief Fusion (GBF)

The generalized belief fusion algorithm \([7]\) depends on probability proportionality weighting functions \(\rho\). The choice of this function is dictated by the nature of the individual belief data set. For any belief data set, \(\rho\) can be chosen to be: \(\rho_j = \{1, |A_j|, \Pi_j, \text{Pr}A_j\} \) with the constant function “1”, the cardinality of \(A\), \(|A|\), the plausibility, \(\Pi\), and proportional to all plausibilities, \(\text{Pr}A\). Special care must be used for \(\rho = \{\text{Bel}\} \) since it may give erroneous results if used with a non-mature belief data set. Therefore, the probability proportionality function \(\rho\) weighting functions can have the following functional values:

\[
\rho_j = \{1, |A_j|, \Pi_j, \text{Pr}A_j, \text{Bel}_j\} \tag{5}
\]

The new methodology of combining independent Multi-Source Beliefs introduced in 2002 is used to fuse \(N\) independent component sets \(\{m_1, m_2, \ldots, m_N\}\) of basic belief assignments (BBAs):

\[
m_{12\ldots N}(A) = \sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B) m_2(C) \ldots m_N(Z) \left( \frac{f(A; B; C; \ldots; Z)}{\Delta} \right)
\]

\[
f(A; B; C; \ldots; Z) = \left[ \frac{C^A[\rho_1]C^B[\rho_2]C^C[\rho_3]}{C^A[\rho_1]C^B[\rho_2]C^C[\rho_3]} \right] \left[ \frac{C^B[\rho_2]C^C[\rho_3]C^D[\rho_4]}{C^B[\rho_2]C^C[\rho_3]C^D[\rho_4]} \right] \ldots
\]

\[
\ldots \left[ \frac{C^Z_1[\rho_N]C^Z_2[\rho_N]}{C^Z_1[\rho_N]C^Z_2[\rho_N]} \right]
\]

(6)

Throughout this article, \(\Delta\) is a normalizing factor chosen so that the sum of BBAs or Pignistic probabilities is one.

The above equations can be generalized to the following form:

\[
m_{12\ldots N}(A; \rho_1, \ldots, \rho_N) = \frac{1}{\Delta} \sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B); \rho_1) \ldots m_N(Z_m(Z); \rho_N) \left( \frac{f(A; B; C; \ldots; Z)}{\Delta} \right)
\]

\[
f(A; B; C; \ldots; Z) = \left[ \frac{C^A_m[\rho_1]C^B_m[\rho_2]C^C_m[\rho_3]}{C^A_m[\rho_1]C^B_m[\rho_2]C^C_m[\rho_3]} \right] \left[ \frac{C^B_m[\rho_2]C^C_m[\rho_3]C^D_m[\rho_4]}{C^B_m[\rho_2]C^C_m[\rho_3]C^D_m[\rho_4]} \right] \ldots
\]

\[
\ldots \left[ \frac{C^Z_m[\rho_N]C^Z_m[\rho_N]}{C^Z_m[\rho_N]C^Z_m[\rho_N]} \right]
\]

(7)

The above formulae provide a straightforward description of the Multi-Source Belief fusion algorithm. The formula fuses all BBAs at once. A much more useful and computational friendly form combines two independent multi-source beliefs at a time. It is the recommended form for implementation:

\[
m_{12\ldots N}(A; \rho_1, \ldots, \rho_N) = \frac{1}{\Delta} \sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B); \rho_1) \ldots m_N(Z_m(Z); \rho_N) \left( \frac{f(A; B)}{\Delta} \right)
\]

\[
f(A; B) = \left[ \frac{C^A_m[\rho_1]C^B_m[\rho_2]}{C^A_m[\rho_1]C^B_m[\rho_2]} \right] \left[ \frac{C^B_m[\rho_2]C^C_m[\rho_3]}{C^B_m[\rho_2]C^C_m[\rho_3]} \right] \ldots
\]

\[
\ldots \left[ \frac{C^Z_m[\rho_N]C^Z_m[\rho_N]}{C^Z_m[\rho_N]C^Z_m[\rho_N]} \right]
\]

(8)

Note that in the combination process, the pedigree information of each set of BBAs is lost. Saving the probability proportionality function for each member of the singleton set, \(\rho_1, \rho_2, \ldots, \rho_N\) can be used to compute the pedigree pignistic probability transform below.

2.1 Pedigree Pignistic Probability

The Pedigree Pignistic Probability (PrPed) introduced in [2] uses the fused BBAs with the probability proportionality functions, \(\rho_1, \rho_2, \ldots, \rho_N\), to compute a better pignistic probability estimate. The Pedigree Pignistic Probability is computed for each singleton element of \(C \subseteq A\) with \(C \leq \Omega\) for all \(A \in 2^\Omega\).

\[
\text{PrPed}(C) = \sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_{12\ldots N}(A; \rho_1, \rho_2, \ldots, \rho_N) \left( \frac{g(C; A)}{\Delta} \right)
\]

\[
g(C; A) = \left[ \frac{C^A_m[\rho_1]}{C^A_m[\rho_1]} \right] \left[ \frac{C^B_m[\rho_2]}{C^B_m[\rho_2]} \right] \ldots \left[ \frac{C^Z_m[\rho_N]}{C^Z_m[\rho_N]} \right]
\]

(9)

3. Dempster-Shafer Belief Fusion

This section demonstrates that the generalized belief fusion algorithm with \(\rho = \{1\}\) is equivalent to Dempster’s rule of combination for the fused BBA.

Combining two BBAs by using Dempster’s rule of combination yields the fused BBA.

\[
m_{12}(A) = \frac{\sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B)) m_2(Z_m(Z)) \left( \frac{f(A; B; C; \ldots; Z)}{\Delta} \right)}{1 - \sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B)) m_2(Z_m(Z))} \tag{10}
\]

Note the fused BBA is normalized to sum to one.

\[
\sum_{A \in 2^\Omega} m_1 \ast_{DS} m_2(A) = 1 \tag{11}
\]

The same two BBAs are combined via the generalized belief fusion algorithm with \(\rho = \{1\}\).

\[
m_{12}(A) = \frac{\sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B)) m_2(Z_m(Z))}{\Delta} \tag{12}
\]

Calculating \(\Delta\) as the normalizing factor for summing the BBAs to one.

4 The Fixsen-Mahler Modified Dempster-Shafer Fusion

Fixsen-Mahler [2,5] and Fister-Mitchell [4] address the combination of two belief subsystems with a specific cardinality weighting called the Modified Dempster-Shafer Fusion (MDS):

\[
m_{12}^{\text{MDS}}(A) = \frac{\sum_{\mathcal{B} \in \mathcal{C}, \mathcal{T} \subseteq \Delta} m_1(B_m(B)) m_2(Z_m(Z)) \left( \frac{g(C; A)}{\Delta} \right)}{\left( \frac{|A|}{\Delta} \left| \frac{A}{|A|} \right| \left| C \right| \right) \Delta} \tag{13}
\]

Calculating Smets [3] pignistic probability for the MDS set of BBA.
The Pedigree Pignistic Probability with the probability proportionality function, \( \rho = \{1\} \) is equal to a normalized Plausibility function (3).

\[
\text{PrPed} (C) = \text{PrNPl}(C)
\]

**6. Numerical Example for Comparison**

A Generalized Belief Fusion Algorithm has been introduced that allows the combination of independent multi-source beliefs that can make faster and better decisions in systems with belief data. In this example, four independent belief measurements are fused with various probability proportionality functions \( \rho \) weighting functions.

Let \( \Omega = \{A, B, C\} \) be the set of 3 possible outcomes with the following notation for the power set.

- \( A = [1,0,0] \)
- \( B = [0,1,0] \)
- \( C = [0,0,1] \)
- \( AB = [1,1,0] \)
- \( AC = [1,0,1] \)
- \( BC = [0,1,1] \)
- \( ABC = [1,1,1] \)

Four independent probability distributions are calculated from a weighted random process and sorted.

\[
P(1) = (0.424922, 0.400284, 0.174794) \\
P(2) = (0.737657, 0.214377, 0.0479658) \\
P(3) = (0.678012, 0.293108, 0.0288797) \\
P(4) = (0.706332, 0.221611, 0.0720563)
\]

(21)

From the above probability distributions, calculate the BBAs with the Inverse Pignistic Probability Transform (IPPT) using the Generalized Sum Mean [8] for the values of \( s = t = 1 \). The BBAs generated thus can be used by all pignistic probability transforms to estimate probabilities. Therefore, proper comparisons can be made for all combinations of belief theories and Naive Bayesian Fusion.

For a given probability set \( P(A) \) with IPPT values of \( s = 1 \) and \( t = 1 \), the basic belief assignments are calculated as:

\[
\text{mm}(A \mid 1,1) = P(A)/D \\
\text{mm}(A, A) \mid 1,1 = \frac{(P(A) + P(A))/2D}{(2D)} \\
\text{mm}(A, A, A) \mid 1,1 = \frac{(P(A) + P(A) + P(A))/3D}{(3D)} \\
\text{mm}(A, A, A, A) \mid 1,1 = \frac{(P(A) + P(A) + ... + P(A))/N*D}{(N*D)}
\]

(22)

D is calculated analytically as \( D = \frac{N^2 -1}{N} \).

For each PDi the calculated the BBAs are:

\[
\text{mm}[1,0,0] = 0.182109 \\
\text{mm}[0,0,1] = 0.0749118 \\
\text{mm}[1,0,1] = 0.128511 \\
\text{mm}[1,1,1] = 0.142857
\]

\[
\text{mm}[2,0,0] = 0.316139 \\
\text{mm}[0,1,0] = 0.142857 \\
\text{mm}[2,0,1] = 0.128511 \\
\text{mm}[2,1,1] = 0.142857
\]

(23)
The singleton Plausibilities for each set of BBAs are calculated as:

\[ P_{\text{Plmm}}(A_i) = \sum_{A_k \supseteq A_i} m_{mk}(A_k) \]  
(27)

The pignistic probability proportional to normalized plausibility (PrNPl) is computed for each singleton element of \( C \subseteq A \) with \( C \subseteq \Omega \) for all \( A \in 2^\Omega \).

\[ \text{PrNPl}(C) = \sum_{A \ni C} m_{mk}(A) \left( \frac{1}{\Delta} \right) \]  
(29)

Smets Pignistic probability for each set of BBAs is calculated as:

\[ \text{BetPmm}_{mk}(A_i) = \sum_{A_w \ni A_i} m_{mk}(A_w) \left( \frac{1}{\Delta} \right) \]  
(31)

The pignistic probability proportional to all Plausibilities (PrPl) is equal to the belief and a component proportional to the sum of all the plausibilities

\[ \text{PrPl}(A_i) = \text{Bel}(A_i) + \varepsilon \ \text{Pl}(A_i) \]  
with \( \varepsilon = \frac{1}{\sum_{A \ni A_i} \text{Pl}(A_i)} \)  
(33)

Combining two BBAs by using Dempster’s rule of combination yields the fused BBA.

\[ m_{mm12}(A) = \sum_{B \subseteq C \subseteq A} \frac{m_{mm1}(B) \ m_{mm2}(C)}{\Delta} \]  
(38)

From above BBAs compute the pignistic probability PrNPl.

\[ \text{PrNPl}1 = \{0.518664, 0.301034, 0.180302\} \]  
(40)

Combining the fused BBAs with the third belief data input by using Dempster’s rule of combination yields the fused BBA.

\[ m_{mm123}(A) = \sum_{B \subseteq C \subseteq A} \frac{m_{mm12}(B) \ m_{mm3}(C)}{\Delta} \]  
(41)

From above BBAs compute PrNPl.

\[ \text{PrNPl}12 = \{0.518664, 0.301034, 0.180302\} \]  
(40)

Combining the fused BBAs with the fourth belief data input by using Dempster’s rule of combination yields the fused BBA.

\[ m_{mm1234}(A) = \sum_{B \subseteq C \subseteq A} \frac{m_{mm123}(B) \ m_{mm4}(C)}{\Delta} \]  
(42)

From above BBAs compute PrNPl.

\[ \text{PrNPl}1234 = \{0.639386, 0.255509, 0.105306\} \]  
(43)

6.2 The Fixsen-Mahler Modified Dempster-Shafer (MDS) Belief Fusion

Combining two BBAs by using the MDS combination rule gives the fused BBAs.
From above BBAs compute Smets Pignistic probabilities:

\[
\text{From above BBAs compute the pedigree Pignistic probabilities with cardinality weighting:}
\]

\[
PrPed12(1, 0, 0) = 0.592948 \quad PrPed12(0, 1, 0) = 0.28066
\]

(48)

Combining all four inputs by using MDS combination rule gives the fused BBAs.

\[
\text{Combining all four inputs by using MDS combination rule gives the fused BBAs.}
\]

\[
nm12^3(A) = \sum_{B \in A} nm1^2(B) \cdot nm2^3(C) \left( \frac{A}{\|B\| C} \right)
\]

\[
nm123[1, 0, 0] = 0.755522 \quad nm123[0, 1, 0] = 0.558438
\]

(49)

From above BBAs compute Smets Pignistic probabilities:

\[
\text{From above BBAs compute the pedigree Pignistic probabilities:}
\]

\[
PrPed123(1, 0, 0) = 0.802357 \quad PrPed123(0, 1, 0) = 0.012662
\]

(51)

Combining all four inputs by using MDS combination rule gives the fused BBAs.

\[
\text{Combining all four inputs by using MDS combination rule gives the fused BBAs.}
\]

\[
nm1234(A) = \sum_{B \in A} nm1^2(B) \cdot nm2^3(C) \cdot nm3^4(D)
\]

\[
nm1234[1, 0, 0, 0] = 0.843472 \quad nm1234[0, 1, 0, 0] = 0.017403
\]

(52)

From above BBAs compute Smets Pignistic probabilities:

\[
\text{From above BBAs compute the pedigree Pignistic probabilities:}
\]

\[
PrPed1234(1, 0, 0, 0) = 0.134758 \quad PrPed1234(0, 1, 0, 0) = 0.00426795
\]

\[
PrPed1234[1, 0, 1, 0] = 0.002226791
\]

(53)

6.3 The Sudano Generalized Belief Fusion with Cardinality Weighting

The Generalized Belief Fusion is calculated with cardinality weighting:

\[
nm12^2(A) = \sum_{B \in A} \left( \frac{\text{nm1}(B) \cdot \text{nm2}(C)}{\|B\| C} \right) \left( \frac{A}{\|B\| C} \right)
\]

\[
nm12[1, 0, 0] = 0.402049 \quad nm12[0, 1, 0] = 0.165792
\]

\[
nm12[0, 0, 0] = 0.0455477 \quad nm12[1, 1, 0] = 0.163999
\]

\[
nm12[1, 0, 1] = 0.113065 \quad nm12[0, 1, 1] = 0.0544129
\]

\[
nm12[1, 1, 1] = 0.0462351
\]

(54)

From above BBAs compute the pediatric Pignistic probabilities with cardinality weighting:

\[
\text{From above BBAs compute the pedestrian Pignistic probabilities with cardinality weighting:}
\]

\[
PrPed12[1, 0, 0] = 0.592948 \quad PrPed12[0, 1, 0] = 0.28066
\]

\[
PrPed12[0, 0, 1] = 0.126392
\]

(56)
From above BBAs compute the Pedigree Pignistic probabilities:

\[ \text{PrPed}1234(0, 0, 1) = 0.924119 \]

\[ \text{PrPed}1234(0, 1, 0) = 0.0545805 \]

(68)

### 6.5 The Sudano Generalized Belief Fusion with PraPl Weighting

The probability proportionality function \( \rho \) has the PraPl weighting function in the Generalized Belief Fusion Algorithm.

\[
\text{PrPed}_1(\text{Pl}) = \sum_{A \in \Theta} \left( \sum_{B \in \Theta} \sum_{C \in \Theta} \frac{m_{AC}(B) m_{AC}(C)}{D(A)} \right) \left( \frac{C_{AC}(\text{Pl}(A))}{C_{AC}(\text{Br}(B))} \right)
\]

(69)

From above BBAs compute the Pedigree Pignistic probabilities:

\[ \text{PrPed}12[0, 0, 1] = 0.768926 \]

\[ \text{PrPed}12[0, 1, 0] = 0.201057 \]

(78)

Fusing the first three inputs:

\[ \text{PrPed}123[0, 0, 1] = 0.893263 \]

\[ \text{PrPed}123[0, 1, 0] = 0.105719 \]

(80)

Fusing all four inputs:

\[ \text{PrPed}1234[0, 0, 1] = 0.964089 \]

\[ \text{PrPed}1234[0, 1, 0] = 0.0357992 \]

(82)

Figure 1 shows the comparison of five methods of fusing the same four independent belief measurements; the probability increase of the most probable state is shown. Note, fusing all four data inputs for the most probable state with D-S theory gives a probability estimate of 0.753, with MDS theory gives a probability estimate of 0.854, with GFB having \( \rho_J = \{\text{Pl}_J\} \) gives a probability estimate of 0.924, with GFB having \( \rho_J = \{\text{PraPl}_J\} \) gives a probability estimate of 0.948, and with GFB having \( \rho_J = \{\text{Bel}_J\} \) gives a probability estimate of 0.964.
8. Conclusion

This article outlined a generalized Belief fusion methodology for making faster and better decisions using Belief function theories. This methodology can be used to automate critical decisions in complex systems. For a given body of belief evidential data, a processing objective is to glean the most correct knowledge from such data (but not more). The author introduced a new methodology of combining independent multi-source beliefs in 2002. This generalized belief fusion algorithm, depending on the probability proportionality weighting functions used, for probability estimates, has been shown to be equivalent to the Dempster–Shafer (DS) theory of evidence, the Modified Dempster–Shafer (MDS) theory, and other fusion methodologies that will converge faster to correct results. A more computationally friendly representation of the generalized belief fusion algorithm is given. An example has been given demonstrating these principles. A reasonable conclusion is to use GBF~PraPl since the results are robust for any type of available belief data while giving good results for making faster and better decisions.

References


