CRLB with Pd<1 for Bearings Only Fused Tracks

Alfonso Farina  
AMS, Rome (Italy)  
afarina@amsjv.it.

Sandro Immediata  
AMS, Rome (Italy)  
simmediata@amsjv.it

Branko Ristic  
Defence Science and Technology Organisation,  
Edinburgh (Australia)  
Branko.Ristic@dsto.defence.gov.au

Luca Timmoneri  
AMS, Rome (Italy)  
ltimmoneri@amsjv.it

Abstract - This paper presents mathematical procedures to calculate the Cramer-Rao Lower Bound (CRLB) for a target track obtained by fusing two tracks provided by two sensors that build up their tracks on the basis of bearings only measurements. It is assumed that the two sensors could have a detection probabilities Pd less than 1. The methods can be easily extended to the fusion of more than two sensor tracks.

1 Introduction

The problem of target tracking by exploiting only the measurements of target bearing has been described in the specialized technical literature, see for instance [1]. It plays a key role in a number of surveillance systems: (i) mobile platform equipped with radar operating in passive mode, (ii) electronic warfare devices (like the electronic support measures, ESM) and (iii) passive sonar.

This paper deals with the problems of:

- fusing the tracks of two sensors measuring the target bearings, and
- computing the Cramer-Rao Lower Bound (CRLB) of the target position and velocity accuracies achievable by fusing the two tracks when the detection probability of the sensors is less than 1.

The Cramer-Rao bounds for nonlinear stochastic filtering [2], [3] (some times referred to as Posterior CRLB-PCRLB) have been derived with an implicit assumption that the measurement sensor is operating with the probability of detection, Pd, equal to unity. Since in many applications - such as target tracking - this is not the case, in this paper we consider the theoretical CRLB for the case where Pd<1. The detection event given by a false alarm is not considered because the probability of false alarm is much smaller than the detection probability (in a typical radar system Pd=0.9 and Pfa=10^{-6}). Two methodologies, Enumeration and IRF (Information Reduction Factor), have been widely described in [4], [5], [6], [7] for the computation of CRLB when Pd<1.

The enumeration solution for the CRLB when Pd<1 is based on the evaluation of an exponentially growing number of possible miss/detection sequences. As the number of sensor scans grows, it becomes more difficult (practically impossible) to compute the exact theoretical bound. Therefore in [4] a feasible approximation has been proposed and verified by simulations in the linear and non-linear stochastic filtering cases. The alternative simpler approach, named IRF, has also been considered and simulated. The implementation of IRF simply requires a division of the sensor variance by the sensor Pd. In this paper both procedures are tested in a simulated environment and the comparison with the ideal case of Pd=1 is also presented.

2 Problem Description

2.1 CRLB computation for one-sensor bearings-only tracking with Pd=1

The moving platform carrying the sensor is called own ship in the following. It is known that the target range is observable only after the own-ship has executed a suitable maneuver [1]. The need for an own ship maneuver distinguishes the bearings-only from the more conventional localization (e.g. classical triangulation) and tracking procedures.

Let the target and the observer state vectors be denoted as \( x^t_k \) and \( x^o_k \) respectively. The own-ship state vector is assumed to be perfectly known. The relative state vector

...
is then defined as \( x_k = x_k^1 - x_k^0 \). For simplicity we adopt the Cartesian coordinates (cc) where \( x_k \) is defined as [5]:

\[
x_k = \begin{bmatrix} x_k & \dot{x}_k & \gamma_k & \dot{\gamma}_k \end{bmatrix}
\]  

(1)

where \([X_k \ \gamma_k] \) denote the relative target position components and \([\dot{X}_k \ \dot{\gamma}_k] \) its velocity components in the \( \chi - \gamma \) plane. Assume that the target is moving with a constant velocity and with zero process noise. Then the state dynamic equation is linear and can be expressed as:

\[
x_{k+1} = F_k x_k + U_{k+1} \]  

(2)

where:

\[
F_k = \begin{bmatrix} 1 & T_k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad U_{k+1} = \begin{bmatrix} x_k^0 - x_k^0 - \dot{x}_k^0 \\ x_k^0 - x_k^0 - \dot{x}_k^0 \\ x_k^0 - x_k^0 - \dot{x}_k^0 \\ x_k^0 - x_k^0 - \dot{x}_k^0 \end{bmatrix}
\]

and \( T_k \) is the time interval between the bearing measurements. Vector \( U_{k+1} \) accounts for the effect of a mismatch between the observer and the target motion models and depends exclusively on own-ship states \( x_k^o \) and \( x_k^o + 1 \).

The angular measurements are collected with \( \text{Pfa}=0 \) and \( \text{Pd}<1 \). Assuming the clockwise positive convention we have:

\[
z_k = h_k(x_k) + v_k
\]

(3)

where:

\[
h_k(x_k) = \arctan\frac{\dot{X}_k}{\dot{\gamma}_k}
\]

(4)

\( v_k \) is zero-mean white Gaussian measurement noise with variance \( R_k = \sigma_v^2 \). In order to compute the CRLB we need to work out the Jacobian of \( h_k(\cdot) \). It can be shown that it results to be [5]:

\[
H_k = \begin{bmatrix} \frac{\gamma_k}{\chi_k^2 + \gamma_k^2} & 0 & -\frac{x_k}{\chi_k^2 + \gamma_k^2} & 0 \end{bmatrix}
\]  

(5)

Since there is no process noise in state equation (2), the calculation of CRLBs is based on a simplified version of the recursive equation, which is described in [2]. The CRLB of component \( j \) of the state vector \( x_k \) is computed as:

\[
C_k[j] = [J_k[j, j]]^{-1},
\]

(6)

where \( J_k \) is the information matrix computed recursively as [2]:

\[
J_{k+1} = [F_k J_k^{-1} F_k^T]^{-1} + H_k^T R_k^{-1} H_k + I_k.
\]

(7)

Prior distribution \( p(x_0) \), which is required to initiate the recursions for the CRLB computation, is assumed to be Gaussian with the covariance matrix given by:

\[
C_0 = \begin{bmatrix} \sigma_x^2 & 0 & \sigma_{x\gamma}^2 & 0 \\ 0 & \sigma_x^2 & 0 & 0 \\ \sigma_{x\gamma}^2 & 0 & \sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_y^2 \end{bmatrix}
\]

(8)

Thus, we end up with the CRLB of the track provided by the sensor, i.e.: the best covariance matrix \( P \) of the target track.

### 2.2 Fusion of tracks with \( P_0=1 \)

A number of strategies have been conceived for fusing information (plots and/or tracks) generated by \( N \) sensors. In the case of \( N \) tracks, they are combined in order to achieve a single multisensor track for each target. This is performed after having associated the corresponding tracks by resorting to a statistical test (see [8]). If the tracks are independents, they are merged into a single equivalent track as follows (see [8] page 17).

\[
\hat{s} = \hat{P} \sum_{i=1}^{N} [\hat{p}_i]^{-1} \hat{s}_i
\]

(9)

\[
\hat{P} = \left\{ \sum_{i=1}^{N} [\hat{p}_i]^{-1} \right\}^{-1}
\]

(10)

where:
\[ \hat{S} = \text{state vector of multisensor track}, \]
\[ \hat{P} = \text{covariance matrix of } \hat{S}, \]
\[ \hat{S}_i = \text{state vector of } i\text{-th monosensor track}, \]
\[ \hat{P}_i = \text{covariance matrix of } \hat{S}_i. \]

A schematic drawing of this fusion algorithm is shown in figure 1 for \( N=2 \), where \((\theta_m)_{1,2}\) represents the target measurements from the two sensors.

![Diagram](image-url)

Figure 1. Multi-sensor filtering through combination of tracks.

In particular, equation (10) provides the CRLB of the fused track. In the case in which the tracks are dependent, the fusion algorithm is the one described in [9]. In the following we will consider only the case of independent tracks.

### 2.3 CRLB computation for fused tracks with \( P_d < 1 \)

This section is dedicated to the computation of the CRLB in case of two sensors which have their own \( P_d < 1 \), \( i=1,2 \). To this aim the following two steps are required:

- the covariance matrix \( P_i \) of the track provided by \( i\text{-th} \) sensor is modified to account for the sensor \( P_d < 1 \), this is done by either resorting to the enumeration method or to the IRF;
- equation (10) is calculated by inserting \( P_i \) determined under the condition of \( P_d < 1 \), \( i=1,2 \).

### 3 Achieved Results

#### 3.1 Simulation software

User-friendly simulation software, implemented using MATLAB © package, has been coded for the scenario generation and the computation of the CRLB. The I/O interface with the software has been realized by means of two panels presented in figures 2 and 3. The first panel is pertinent to the scenario generation. Two own-ships can be identified which may or may not maneuver; the initial conditions (position and velocity in the own ship reference system) can be suitably set. Two targets can be managed with a set of possible maneuvers; other input data are the sensor data rate, the total number of scans and the timing of the two sensors.

![Diagram](image-url)

Figure 2. Scenario generation panel.

A second panel has been defined for the CRLB computation (figure 3). It allows the computation of the CRLB for one own-ship case and the CRLB for the two sensors case calculated with the Enumeration and IRF methods. The other inputs are the \( P_d \), the sensor accuracy and a suitable threshold, which defines the maximum number of miss/detections to include in the implementation of the enumeration method [4].

The flexibility of the code allowed us to run the software for a number of different scenarios. Next sections 3.2 and 3.3 report the results achieved in two scenarios. The first one is pertinent to the fusion of two sensor tracks in presence of one target; for this case, several outputs are presented. The second scenario concerns with the more complex case of two own-ships and two targets; only the CRLB will be shown.
3.2 First Scenario

Purpose of this first study case is the evaluation of the CRLB for a system constituted by two sensors. The tracking results are fused in the case of $P_d=90\%$ for each sensor.

The CRLB has been evaluated using the enumeration and IRF techniques. The hypothesized scenario is depicted in figure 4.

![Diagram](image)

**Figure 3. CRLB panel.**

The sensors are mounted on board of two airplanes performing uniform circular motion with centripetal acceleration of 0.5 g. The velocity amplitude is equal to 200 m/s. The first airplane has a clockwise motion, while the second is moving in counter-clockwise direction.

The target is moving along the straight line between the two sensors; its velocity components are: $v_x$ (velocity along x-axis)=150 m/s, $v_y$ (velocity along y-axis)=120 m/s.

It has been assumed that the two sensors have the same angular accuracy, $\sigma_\alpha$, equal to 1°. Data rate is 30 seconds for each sensor and the temporal interval between the plots acquired by the two sensors is 15 seconds, i.e. the plots are interleaved with period equal to 15 seconds.

The covariance matrix of the estimation, $P_{k|k}$, is computed for the enumeration technique with a maximum number of miss-detections equal to 7 out of 13 possible detections; these values guarantee an accuracy of 0.99 for the evaluation of the CRLB.

The achieved accuracies in position and velocity versus time are reported respectively in figures 5 and 6 for the first sensor (on board of the platform with clockwise motion) while figures 7 and 8 present the same quantities for sensor 2 (the one with counter clockwise motion). Each figure presents six curves; for figures 5 and 7 three curves are pertinent to the position accuracies along x axis and the other three curves relate to the position accuracies along y axis. Following the same strategy, figures 6 and 8 compare the velocity accuracies achieved for each sensor. The accuracies are calculated by applying the three methodologies: (i) enumeration with $P_d=0.9$, (ii) IRF with $P_d=0.9$, and (iii) $P_d=1$.

![Diagram](image)

**Figure 5. Tracking position results (sensor 1).**
Figure 6. Tracking velocity results (sensor 1).

Figure 7. Tracking position results (sensor 2).

Figure 8. Tracking velocity results (sensor 2).

Figure 9. Comparison between position accuracy calculated with enumeration criteria.

Figures 10 and 11 depict the results of the fusion of the two tracks. As usual, the CRLB of the fused track computed with the enumeration method is compared with the same quantity computed via IRF and with the case of $P_d=1$. It is seen that the IRF method gives an optimistic evaluation of the tracking accuracy.

Figure 10. Tracking position result of fusion along x.

Figure 11. Tracking position result of fusion along y.

Figure 9 shows that, due to the different positions of the two sensors with respect to the target, the achieved tracking accuracies by the two sensors are different; more in detail, the first sensor has a worse position accuracy along the x-axis and a better accuracy along the y-axis.
Figure 11. Tracking position result of fusion along y.

Figure 12 compares the areas of the ellipsoid of uncertainty (99% of probability for the estimated target position to fall inside the ellipsoid area) computed by means of the CRLB using the enumeration technique and for the two sensors each with a data rate of 15 seconds (i.e. half of the data rate considered for preceding figures). This new value of data rate for the two sensors is motivated by the need of comparing the fused track with two “single” tracks having the same number of plots of the fused one.

We note that the fused track gives better accuracy results if the time of observation is enough long (150 seconds for the case under analysis). The improvement found is also related to the fact that the two sensors have roughly the same position and velocity accuracies.

3.3 Second scenario

The two own-ship characteristics are the same described in previous section 3.2 except for the following: the first sensor has a counter-clockwise motion and the second sensor shows a clockwise motion. The two simulated targets parameters are: Target 1. linear, constant velocity motion with velocity components: \( v_x = 180 \text{ m/s}, \ v_y = -50 \text{m/s} \). Target 2. linear, constant velocity motion with velocity components: \( v_x = 180 \text{ m/s}, \ v_y = 50 \text{m/s} \).

The scenario is depicted in figure 13. It is assumed that the plot track correlation logic performs perfectly.

The results achieved in the computation of the CRLB of the target position estimation after the tracks fusion is reported in figure 14 for the first target and in figure 15 for the second target. The CRLB is compared with the similar bound computed for the single sensor case; as for the first scenario, the data rate for the single sensor case is twice the data rate of the two sensors considered for the fused tracks. So doing, the two study cases (fused/no fused) are fairly compared. The benefit of the sensor fusion is evident.

We note that the fused track gives better accuracy results if the time of observation is enough long (150 seconds for the case under analysis). The improvement found is also related to the fact that the two sensors have roughly the same position and velocity accuracies.
4 Conclusions

The CRLB in case of fusing the tracks pertaining to two bearings-only sensors has been computed for the case of Pd<1. The Enumeration and IRF methods have been introduced and used for the analysis. The benefit in terms of improvement of the tracker accuracy after the fusion has been clearly demonstrated. It is noted that the IRF methods is too optimistic even though very simple from a computational point of view. In the future we plan to compare the CRLB results with Monte Carlo simulation. This study is worth to do because bias problems are known to occur for non-linear tracking filters for passive ranging applications.

References


