Abstract – In Level 2 Fusion, the concept of data association is much broader than in the Level 1 case. Apart from the concept of associating objects to one another, in Level 2, how the combination of lower level objects associate with interpretations of the scene is also important. One interpretation of this view is that the association among objects is performed through nonlinear and symbolic functions and that the solution can be assigned a symbolic interpretation. To this end, we must create metrics or measures among objects and among objects and abstraction of the scene. In this paper, we introduce several mathematically based metrics for the association in the Level 2 problem. These measures have been developed using mathematical rigor.

Keywords: Level 2 Fusion, data association, metrics, discrete metric, continuous metric.

1 Introduction

As defined in [6], Level 2 Fusion is the estimation and prediction of relations among entities, which can be referred to as Situational Assessment. In recent work [9], the authors proposed a general architecture for the Level 2 problem. As expressed in Figure 1, the basic approach to the Level 2 problem mimics that of the Level 1 problem. The difference between the two approaches is in the interpretation and implementation of each of the components. While some may feel that this architecture is just an extension of a Kalman filter and others ([1], [3], and [4]) have shown that association need not be an essential part of the fusion process, this architecture provides a firm foundation for and understanding of Level 2 implementations and is not limited to a particular estimation approach.

Figure 1. The proposed architecture allows the Level 2 Fusion problem to be decomposed into smaller and more easily addressable subproblems.

In this paper, we address the issue of association, which we consider an integral part of our Level 2 Fusion approach. The foundation of the association is the ability to provide a comparative measure to rank the possible combinations. While the metrics that we have pursued in our effort were originally conceived for a military ground target problem, they are generic enough in nature to be applied to other fusion problems.

In Section 2 of this paper, we provide the mathematical definition of a metric. In Section 3, we introduce two metrics proposed for discrete elements, while in Section 4, we introduce a continuous value measure. In Section 5, we outline the approach to create what we refer to as a joint metric, which is used to provide the necessary single value when two or more measures are used for comparison.

2 Mathematical Metrics

In order to provide a strong theoretical foundation for our work, we start with a brief discussion of metric spaces and the cardinality function that lies at the heart of the discrete metric. The metric space discussion is terse and at a high level. A more detailed discussion can be found in most elementary analysis or topology books [2]. We conclude this section with a definition of the cardinality function and some of the required properties for the development of the discrete metric function.

2.1 Metric spaces

This section lays the mathematical foundation for the development of our Level 2 Fusion work. We begin with the definition of a metric space.

Definition 2.1: A pair of objects \((X, d)\) consisting of a nonempty set \(X\) and a function \(d: X \times X \rightarrow \mathbb{R}\), where \(\mathbb{R}\) is the set of real numbers, is called a metric space provided that

\[
\begin{align*}
&d(x, y) \geq 0, \quad x, y \in X; \\
&d(x, y) = 0 \quad \text{if and only if} \quad x = y, \quad x, y \in X; \\
&d(x, y) = d(y, x), \quad x, y \in X; \\
&d(x, y) \leq d(x, z) + d(z, y), \quad x, y, z \in X.
\end{align*}
\]

The function \(d\) is called a distance function or metric on \(X\) and the set \(X\) is called the underlying set.

Throughout our development of distance metrics for Level 2 Fusion, it is necessary to be able to combine multiple distance metrics into a single distance metric.
The next two theorems describe common schemes for accomplishing this feat.

**Theorem 2.1:** Let \( d_i \) and \( d_j \) be metric spaces on the underlying set \( X \), then the distance function defined by \( d(A, B) = \max \{d_i(A, B), d_j(A, B)\} \) is also a metric space on the underlying set \( X \).

**Proof:** We need to verify the four conditions of the metric space.

(a) Since \( d_i(A, B) \geq 0 \) and \( d_j(A, B) \geq 0 \), we have \( \max \{d_i(A, B), d_j(A, B)\} \geq 0 \) so that \( d(A, B) \geq 0 \).

(b) Suppose \( d(A, B) = 0 \), then by definition \( d_i(A, B) = 0 \) and \( d_j(A, B) = 0 \), but these are metric spaces so \( A = B \). Conversely, suppose that \( A = B \), then \( d_i(A, B) = 0 \) and \( d_j(A, B) = 0 \) since they are metric spaces so that we have \( d(A, B) = 0 \).

(c) For symmetry, we have the following
\[
\begin{align*}
d(A, B) &= \max \{d_i(A, B), d_j(A, B)\} \\
&= \max \{d_i(B, A), d_j(B, A)\} \\
&= d(B, A)
\end{align*}
\]

(d) Finally, for the triangle inequality, determine the values
\[
\begin{align*}
i_{da} &= \arg \max \{d_i(A, B), d_j(A, B)\} \\
i_{de} &= \arg \max \{d_i(B, C), d_j(B, C)\}
\end{align*}
\]

As a result, \( d(A, B) = d_{i_{da}}(A, B) \) and \( d(B, C) = d_{i_{de}}(B, C) \). Then, we see that
\[
d_i(A, C) \leq d_i(A, B) + d_i(B, C)
\]

But since
\[
d_{i_{da}}(A, B) + d_{i_{de}}(B, C) = d(A, B) + d(B, C)
\]
we arrive at the desired result of
\[
d_i(A, C) \leq d(A, B) + d(B, C).
\]

Similarly,
\[
d_j(A, C) \leq d(A, B) + d(B, C).
\]

Thus, we have
\[
d(A, C) = \max \{d_i(A, C), d_j(A, C)\} \\
\leq \max \{d_i(A, B) + d_i(B, C), d_j(A, B) + d_j(B, C)\} \\
\leq \max \{d(A, B) + d(B, C), d(A, B) + d(B, C)\}
\]
so that \( d(A, C) \leq d(A, B) + d(B, C) \) and have shown that this distance measure is indeed a metric space.

QED

**Theorem 2.2:** Let \( d_i \) and \( d_j \) be metric spaces on the underlying set \( X \), then the distance function defined by \( d(A, B) = d_i(A, B) + d_j(A, B) \) is also a metric space on the underlying set \( X \).

**Proof:** The proof of this theorem follows directly from Definition 3.1 and the fact that \( d_i \) and \( d_j \) are metric spaces.

QED

Although the results given in Theorem 2.1 (maximum of two metrics) and 2.2 (sum of two metrics) are both options for creating joint metrics, we will use the maximum of two metrics to join continuous and discrete metrics for this work.

### 3 Discrete Element Metrics

As we mentioned above, the Level 2 problem that we investigated for our research was a military ground target scenario. While such scenarios provided the impetus for our development, the goal of our research was to develop techniques that extended to general Level 2 problems. With this fact in consideration, we develop the metrics in this paper in general with the description of the applications for military problems.

In this section, we introduce discrete element metrics. The two metrics presented are the cardinality-based metric and the gap metric. For these metrics we assume that a group of Level 1 objects can be decomposed in a finite vector whose elements can be mapped into a countable set of values. For example, the vector can represent the different vehicle classes that compose the group. Each element of the vector contains the number of vehicles in the group of the given class:

\[
\text{Group}_{i} = \begin{bmatrix}
\text{Tanks} \\
\text{APC} \\
\text{Command}
\end{bmatrix} = \begin{bmatrix}
2 \\
5 \\
1
\end{bmatrix}
\]

As for our two metrics, the cardinality-based metric can be considered the distance between two vectors, and the gap can be considered the angle.

#### 3.1 Cardinality and its associated metric

We begin by defining the cardinality function and describe some of the important properties that are required for the development of the discrete metric.

**Definition 3.1:** Let \( A \) be a finite set, then
\[
\text{card}(A) = \begin{cases}
0, & \text{if } A = \emptyset \\
n, & \text{if } A \neq \emptyset \text{ and } n = \# \text{ of elements in } A
\end{cases}
\]
is the cardinality function of the set \( A \). Next, we give some properties and simple results for the cardinality function.
Property 3.1: \( \text{card}(A) = 0 \) if and only if \( A = \emptyset \).

Property 3.2: If \( A \) and \( B \) are disjoint sets, \( A \cap B = \emptyset \), then \( \text{card}(A \cup B) = \text{card}(A) + \text{card}(B) \).

Property 3.3: \( \text{card}(A \cup B) = \text{card}(A) + \text{card}(B) - \text{card}(A \cap B) \).

Lemma 3.1: Let \( A \subseteq B \). Then \( \text{card}(A) \leq \text{card}(B) \).

Lemma 3.2: Let \( A \subseteq B \). Then \( \text{card}(A) = \text{card}(B) \) if and only if \( A = B \).

### 3.1.1 Metric development

The goal of this section is to describe a metric that is based upon the cardinality function defined in the previous section. We will begin our discussion with an unsigned metric that satisfies the conditions in Definition 2.1. We will then provide one possible generalization to a signed metric similar to absolute value for the real line. Finally, we will demonstrate a relationship between the signed and unsigned metric.

Let \( A \) and \( B \) be two discrete (finite element) sets. We propose that a discrete set metric (a function that measures the “distance” between two sets) be defined by

\[
\text{d}(A, B) = \text{card}(A \cup B) - \text{card}(A \cap B).
\]

Given the above definition, we must show that \( d \) is indeed a metric; that is, \( d \) is a metric if it satisfies the criteria set forth in Definition 2.1.

**Theorem 3.1: The function defined by**

\[
\text{d}(A, B) = \text{card}(A \cup B) - \text{card}(A \cap B)
\]

**is a distance metric.**

**Proof:** We must show that the four conditions of a metric space are met:

(a) Since \( (A \cap B) \subseteq (A \cup B) \), we have \( \text{card}(A \cup B) \leq \text{card}(A \cap B) \) so that \( \text{d}(A, B) \geq 0 \).

(b) Let \( A = B \), then

\[
\text{d}(A, A) = \text{card}(A \cup A) - \text{card}(A \cap A) = \text{card}(A) - \text{card}(A) = 0.
\]

Now let \( \text{d}(A, B) = 0 \). Then

\[
\text{card}(A \cup B) = \text{card}(A \cap B).
\]

But \( A \cap B \subseteq A \cup B \), implying (by Lemma 3.2) that \( A \cup B = A \cap B \). Suppose \( A \neq B \). Then either there exist \( e \in A \) such that \( e \notin B \) or there exist \( e \in B \) such that \( e \notin A \). Without loss of generality, assume there exist \( e \in A \) such that \( e \notin B \). Since \( e \in A \), then

\[
e \in A \cup B = A \cap B,
\]

which implies \( e \in A \) and \( e \notin B \), contradicting the assumption that \( e \notin B \). The argument is symmetric for \( e \in B \) such that \( e \notin A \), leading to the contradicting the assumption that \( e \notin A \). Thus, supposing that \( A \neq B \) leads to a contradiction, we must have \( A = B \).

(c) Clearly the defined distance satisfies the symmetry condition.

(d) For the triangle inequality condition, we have

\[
\text{d}(A, C) = \text{card}(A \cup C) - \text{card}(A \cap C)
\]

\[
= \text{card}(a) + \text{card}(ab) + \text{card}(ac) + \text{card}(abc) + \text{card}(bc) + \text{card}(c)
\]

\[
- \text{card}(ac) - \text{card}(abc).
\]

Likewise,

\[
\text{d}(A, B) = \text{card}(a) + \text{card}(ac) + \text{card}(bc) + \text{card}(b)
\]

and

\[
\text{d}(B, C) = \text{card}(b) + \text{card}(ab) + \text{card}(ac) + \text{card}(card).\]

Comparing the individual terms, we clearly see that

\[
\text{d}(A, C) \leq \text{d}(A, B) + \text{d}(B, C).
\]

Finally, the triangle inequality is clearly satisfied.

QED

![Figure 2. Venn diagram showing the possible intersections between three sets.](image-url)
where \( dv \) denotes discrete vector. The vector norm we choose is the \( \ell_1 \)-norm, also referred to as the column sum.

At this point, we now have a discrete metric that can measure the distance between two Level 2 objects based upon their classification and can compare a Level 2 object to templates of known Level 2 objects.

### 3.2 Gap metric

In the one-dimensional or one-object class case, a signed distance measure provides all of the information that determines the distance difference or error between two sets. However, as can be inferred from Figure 3, the cardinality-based metric is seen as a distance or range comparison. Both Group 1 and Group 2 have cardinality-based measure of 1, but they are clearly not the same solution. The ratio of the group composition, or angle between the vectors, is clearly different. However, if we use multiple metrics, we can incorporate more information that will describe the difference between two sets. We considered the angle metric or gap metric.

The canonical angles between subspaces create the gap topology [7], which has several different associated metrics. As shown in [7], both the arc cosine of the angle and the sine of the angle are metrics. The metric we use for this effort is the sine of the major angle (the maximum canonical angle) between the subspaces

\[
\sin \theta_{\text{max}}
\]

or the inverse cosine of the product of the canonical angles

\[
\arccos \left( \prod \cos(\theta) \right).
\]

In our case, the subspaces being compared are single vectors, which, in turn, result in a single angle. To compute the angle between the subspaces for the discrete cardinality measure, in this case single vectors, we use the arc cosine of the inner product of the two vectors that represent two separate objects

\[
\theta = \arccos \left( \frac{x^T y}{(x^T x)^{1/2} (y^T y)^{1/2}} \right)
\]

The resulting angle is defined between 0 and \( \pi \).

### 4 Continuous Metric

While the discrete metric was originally designed for comparing group composition, the continuous metric research was designed to develop a measure of the kinematic distance between two groups. In Figure 4, two groups of Level 1 objects exist. We wish to determine a distance measure.
The metric that we developed to determine this measure is

\[ d_p(A, B) = \max \left\{ \min \left\{ \| a_i - b_j \| : \text{for all } a_i \in A \right\} \right\} \]

and

\[ \min \left\{ \| a_i - b_j \| : \text{for all } b_j \in B \right\} \] so that

\[ d_p(A, B) = \max \{ d_a(A, B), d_b(A, B) \} . \]

Given this notation, we now must verify that each of the four conditions of a metric is satisfied:

(a) Since \( d(a, b) = 0 \) for each \( a \in A \) and \( d(b, A) = 0 \) for each \( b \in B \), we have \( d_p(A, B) \geq 0 \).

(b) Suppose \( A = B \), then for each \( a \in A \), there is a corresponding \( b \in B \) such that \( a = b \) so that \( \| a_i - b_j \| = 0 \) and \( d(a, B) = 0 \). Similarly, for each \( b \in B \), we have \( d(b, A) = 0 \). Therefore, \( d_p(A, B) = 0 \) and we have shown that \( A = B \) implies that \( d_p(A, B) = 0 \). Conversely, if \( d_p(A, B) = 0 \), then we have \( d_a(A, B) = 0 \) and \( d_b(A, B) = 0 \). Select any \( a \in A \), then since \( d(a, B) = 0 \), there exists a corresponding \( b \) such that \( a = b \) so that \( A \subseteq B \). A similar argument gives \( B \subseteq A \) so that \( A = B \). Therefore, we have shown that \( d_p(A, B) = 0 \) implies that \( A = B \).

(c) For symmetry, we must show that \( d_p(A, B) = d_p(B, A) \). Observe that

\[ d_p(A, B) = \max \{ d_a(A, B), d_b(A, B) \} \]

\[ = \max \left\{ \max \{ d(a, B), d_b(A) \}, \max \{ d(a, B), d_a(A) \} \right\} \]

\[ = \max \{ d_a(B, A), d_b(B, A) \} \]

\[ = d_p(B, A) \]

(d) Finally, for the triangle inequality condition, let \( a \in A \) be any element. Then, define the following quantities:

\[ b_j = \arg \min_{b_j \in B} \| a_i - b_j \| \]

\[ c_i = \arg \min_{c_i \in C} \| b_j - c_i \| \]

\[ a_i = \arg \min_{a_i \in A} \| a_i - c_i \| \]

These points are shown in Figure 5. From this, we notice that following relationship holds

\[ d(a, C) = \min_{c \in C} \| a_i - c \| = \| a_i - c_i \| \leq \| a_i - c_i \| . \]

Then, we have the following
\[ d_\alpha(A, C) = \max_{i, k} \| a_i - c_k \| \]
\[ \leq \max_{i, k} \| a_i - c_k \| \]
\[ \leq \max_{i, k} \{ \| a_i - b_i \| + \| b_i - c_k \| \} \]
\[ \leq \max_{i, k} \| a_i - b_i \| + \max_{i, k} \| b_i - c_k \| \]
\[ = d_\alpha(A, B) + \max_{i, k} \| b_i - c_k \| \]
\[ \leq d_\alpha(A, B) + d_\beta(B, C) \]
\[ \leq d_\alpha(A, B) + d_\beta(B, C) \]

where the second to last inequality is the result of the fact that collection of elements from \( B \) that are minimums for elements in \( A \) is a subset of the entire collection of elements in \( B \). A similar argument gives

\[ d_\beta(A, C) \leq d_\alpha(A, B) + d_\beta(B, C) \]

so that we arrive at

\[ d_\alpha(A, C) \leq d_\alpha(A, B) + d_\beta(B, C) \, . \]

QED

Figure 5. Illustration of relationships for verification of the triangle inequality property.

5 Development of a Joint Metric

At this point, we now have demonstrated a continuous metric as well as two discrete metrics. What remains to be discussed is how these metrics are combined into a single joint metric. This section accomplishes this task.

In Section 3, we discussed the cardinality vector, and developed a metric for the multidimensional problem. If we look at the use of multiple metrics as a similar problem, we can approach the resolution again with a vector norm. In this section, we propose a two-step process. First we develop the joint discrete metric and then an overall joint metric.

The two discrete metrics form a 2-tuple

\[ (d_\alpha(A, B), d_\beta(B, C)) \]

\[ d_\alpha(A, B) = \begin{bmatrix} \text{metric(}\theta\text{)} \\ \| d_{\alpha}(A, B) \| \end{bmatrix} \]

where \( \text{metric(}\theta\text{)} \) is either \( \sin\theta \) or \( \theta \). This vector represents the distance and angle between the vectors being compared. A weighted \( \ell_2 \)-norm of the vector is used

\[ d_\gamma(A, B) = \left( \alpha \| d_{\alpha}(A, B) \|^2 + \beta \| \text{metric(}\theta) \|^2 \right)^{1/2} \]

Lemma 5.1: The \( \ell_2 \)-norm of the vector (3) with the gap metric chosen as \( \theta \) is also a metric of the discrete vector representations.

Proof: The first condition \( d(x, y) \geq 0 \), \( x, y \in X \) is met from the definition of the norm and the fact that the square of \( \sin x \) is always positive.

The second condition \( d(x, y) = 0 \) if and only if \( x = y \), \( x, y \in X \) can also easily be shown. First, the norm in (4) is only zero if and only if all of the elements are zero. Since both elements are computed via metrics, they can only be zero if and only if \( x = y \).

The third condition, \( d(x, y) = d(y, x) \), \( x, y \in X \), again follows from the fact that the components of (4) are metrics. Since the elements do not change based on the order of the function, Condition 3 is met.

Condition 4,

\[ d(x, y) \leq d(x, z) + d(z, y) \], \( x, y, z \in X \), is also satisfied. From Eq. 4 we get

\[ d_\gamma(A, C) = \left( \| d_{\alpha}(A, C) \|^2 + \sin^2 \theta_{AC} \right)^{1/2} \]

and now show the inequality

\[ d_\gamma(A, B) = \left( \| d_{\alpha}(A, B) \|^2 + \sin^2 \theta_{AB} \right)^{1/2} \leq \]

\[ \left( \| d_{\alpha}(A, C) \|^2 + \sin^2 \theta_{AC} \right)^{1/2} + \left( \| d_{\alpha}(C, B) \|^2 + \sin^2 \theta_{CB} \right)^{1/2} \]

We are able to prove this if we can show

\[ \| d_{\alpha}(A, B) \|^2 + \sin^2 \theta_{AB} \leq \]

\[ \left( \| d_{\alpha}(A, C) \|^2 + \| d_{\alpha}(C, B) \|^2 + \sin \theta_{AC} + \sin \theta_{CB} \right)^2 \]

\[ \Leftrightarrow \| d_{\alpha}(A, B) \|^2 + \sin^2 \theta_{AB} \leq \left( R + \sin \theta_{AC} + \sin \theta_{CB} \right)^2 \]

We note that \( \| d_{\alpha}(A, B) \| \leq R \) from the fact that the elements of \( d_{\alpha}(\bullet) \) are metrics. Since the elements of the
summed vectors are larger or equal to all of the elements of the single vector. This implies that the norm of the summed vector is greater than or equal to the single vector. In turn, the sum of the normed individual components is greater than or equal to the norm of the combined vector.

We know that \( \sin \theta_{AB} \leq \sin \theta_{AC} + \sin \theta_{CB} \) from the fact that \( \sin \theta \) is metric for the canonical angle. Since both elements of the right hand side of (3.8) are solutions to metrics they are positive, we can write

\[
\|d_{\theta}(A,B)\|_2^2 + \sin^2 \theta_{AB} \leq R^2 + \left( \sin \theta_{AC} + \sin \theta_{CB} \right)^2 \\
\left( R + \sin \theta_{AC} + \sin \theta_{CB} \right)^2
\]

From the properties of norms, we can write

\[
d_{\theta}(A,B)^2 \leq \left( R + \sin \theta_{AC} + \sin \theta_{CB} \right)^2
\]

\[
\leq \left( \left( \|d_{\theta}(A,C)\|_2^2 + \sin^2 \theta_{AC} \right)^{1/2} + \left( \|d_{\theta}(C,B)\|_2^2 + \sin^2 \theta_{CB} \right)^{1/2} \right)^2
\]

QED

A similar proof can be constructed for the case where the gap metric is set to \( \theta \). The weighted metric allows us to relate the measures and rank them in order of importance. With this single measure we can combine the continuous metric and the discrete metric. Again, we utilize the weighted \( \ell_2 \)-norm.

The weighting again allows us to combine apples and oranges (continuous measure, discrete distance measure, and discrete angle measure) in a desired relationship as above. The concept of weighting the various elements that compose the metric can be quite important. As in the discussion on weighted least squares in [8], the concept is that the weight changes the concept of length or unit of the components of the vector. Also, weight can incorporate a degree of uncertainty or a degree of importance of each component. We see this same result in the development of the chi-squared metric when measurements of different units and uncertainty are given, such as in the case of the range-bearing type measurement.

6 Conclusions

In this paper, we introduce several metrics to be applied to the Level 2 fusion problem. These metrics have already been used in test cases for development of an automated Level 2 Fusion system applied to ground target scenarios. These metrics only represent the beginning of the investigation in this area. At this time metrics to assist in determining formation and region of influence have also been developed.

With this research, we have made small steps into the area of general development of algorithms and approaches for Situation Assessment.

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References


